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SUMMARY

An iteration method is employed to obtain the flow of a compressible fluid past a curved surface. The first approximation, which leads to the Prandtl-Glauert rule, is based on the assumption that the flow differs but little from a pure translation. The iteration process then consists in improving this first approximation in order that it will apply to a flow differing from pure translatory motion to a greater degree. The method fails when the Mach number of the undisturbed stream reaches unity but permits a transition from subsonic to supersonic conditions without the appearance of a compression shock. The limiting value of the undisturbed stream Mach number, defined as that value at which potential flow no longer exists, is indicated by the apparent divergence of the power series representing the velocity of the fluid at the surface of the solid boundary.

For small Mach numbers and for thin shapes, the results obtained by the iteration process agree with those obtained by the Poggi method. For higher values of the stream Mach number less than the critical value, numerical calculations are in agreement with the results obtained by von Kármán by means of the hodograph method. For values of the stream Mach number higher than the critical value, the iteration process yields some information about the region of flow comprised between the critical stream Mach number and the limiting stream Mach number.

INTRODUCTION

When a body is held fixed in a compressible fluid moving at a uniform speed less than, but comparable with, that of sound, there may be a region near the surface where the velocity of the fluid relative to the body exceeds the local velocity of sound. The flow in such cases may be perfectly regular with no indication of shock waves. Several such types of flow have been described by Taylor

(reference 1) and, more recently, by Görtler (reference 2). In connection with this type of flow it is important to know when the irrotational motion ceases to be possible. It is certain that irrotational motion is no longer possible as soon as the Mach number of the undisturbed stream reaches a definite value, always less than unity, which depends on the shape of the body. In the past compressibility shock has often been assumed to occur when the maximum velocity of the fluid at the surface of the body equals the local velocity of sound; however, the papers of Taylor and Görtler (references 1 and 2) question whether this is correct for the first appearance of a shock wave. In addition, a recent paper by von Kármán (reference 3) suggests that the envelope of the Mach lines in the supersonic region of flow probably introduces the first shock wave in the flow. The stream Mach number at which the envelope of the Mach lines first appears may thus be identified with a limiting value of the Mach number.

The present paper treats the flow of a compressible fluid past a curved surface by means of an iteration process based on that of Ackeret (reference 4). The boundary was so chosen as to conform with the requirements of the method; namely, no stagnation points and small variation of the local velocity from that in the stream. The process, furthermore, permits values of the stream Mach number ranging from zero to the neighborhood of unity. The method proves to be quite laborious when more than two stages in the iteration are demanded; but, because of the importance of the problem, it has been thought worth while to perform the third step. Most of the details of the calculations have been relegated to appendixes in order not to disturb the continuity of the main ideas. (The equations in the appendixes have been assigned numbers prefixed by letters denoting the appendix; for example, equation (A-3) is the third equation in appendix A.)

THE ITERATION PROCESS

The fundamental differential equation governing the flow of a compressible fluid is

$$\left(c^2 - u^2\right)\frac{\partial u}{\partial X} + \left(c^2 - v^2\right)\frac{\partial v}{\partial Y} - uv\left(\frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y}\right) = 0 \qquad (1)$$

where

X, Y rectangular Cartesian coordinates in plane of profile

u, v fluid velocity components along X and Y axes

c local velocity of sound

The condition for irrotational motion is that

$$\frac{\partial u}{\partial Y} = \frac{\partial v}{\partial X}$$

and leads to a velocity potential ϕ' defined by

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial X} \\ v &= \frac{\partial \phi}{\partial Y} \end{aligned}\right\} \tag{2}$$

If the body is held fixed in a uniform stream of velocity U, the relation between the local velocity of sound c and the velocity of the fluid $\sqrt{u^2 + v^2}$ is given for adiabatic processes by

$$\frac{c^2}{c_o^2} = 1 + \frac{\gamma - 1}{2} M^2 \left(1 - \frac{u^2 + v^2}{U^2}\right) \tag{3}$$

where

c_o velocity of sound in undisturbed stream

γ ratio of specific heats at constant pressure and constant volume

M Mach number of undisturbed stream (U/c_o)

With the introduction of a characteristic length s as unit of length and the stream velocity U as unit of velocity, the various quantities thus far defined can be rendered nondimensional. Thus X, Y, u, v, and ϕ denote, respectively, the nondimensional quantities X/s, Y/s, u/U, v/U, and ϕ/Us while c and c_o retain their original meanings. By use of equation (2), equations (1) and (3) then become, respectively,

$$\left(\frac{c^2}{c_o^2} - M^2u^2\right)\frac{\partial^2\phi}{\partial X^2} + \left(\frac{c^2}{c_o^2} - M^2v^2\right)\frac{\partial^2\phi}{\partial Y^2} - 2M^2\frac{\partial\phi}{\partial X}\frac{\partial\phi}{\partial Y}\frac{\partial^2\phi}{\partial X\partial Y} = 0 \qquad (4)$$

and

$$\frac{c^2}{c_o^2} = 1 + \frac{\gamma - 1}{2}M^2\left[1 - \left(u^2 + v^2\right)\right] \qquad (5)$$

Let t denote a characteristic parameter of the shape, such as the thickness coefficient; then, the following expansions are assumed:

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial X} = 1 + t\frac{\partial\phi_1}{\partial X} + t^2\frac{\partial\phi_2}{\partial X} + t^3\frac{\partial\phi_3}{\partial X} + \ldots \\ v &= \frac{\partial\phi}{\partial Y} = t\frac{\partial\phi_1}{\partial Y} + t^2\frac{\partial\phi_2}{\partial Y} + t^3\frac{\partial\phi_3}{\partial Y} + \ldots \end{aligned}\right\} \qquad (6)$$

When these expressions for u and v are introduced into equation (4), together with the expression for c^2/c_o^2 given by equation (5), and when the coefficients of the various powers of t are equated to zero, the following differential equations for ϕ_1, ϕ_2, ϕ_3, . . . result:

$$\left(1-M^2\right)\frac{\partial^2\phi_1}{\partial X^2} + \frac{\partial^2\phi_1}{\partial Y^2} = 0 \qquad (7)$$

$$\left(1-M^2\right)\frac{\partial^2\phi_2}{\partial X^2} + \frac{\partial^2\phi_2}{\partial Y^2} = M^2\left[(\gamma+1)\frac{\partial\phi_1}{\partial X}\frac{\partial^2\phi_1}{\partial X^2} + (\gamma-1)\frac{\partial\phi_1}{\partial X}\frac{\partial^2\phi_1}{\partial Y^2} + 2\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_1}{\partial X\partial Y}\right] \qquad (8)$$

$$(1-M^2)\frac{\partial^2\phi_3}{\partial X^2}+\frac{\partial^2\phi_3}{\partial Y^2}=M^2\left\{\frac{1}{2}\left(\frac{\partial\phi_1}{\partial X}\right)^2\left[(\gamma+1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_1}{\partial Y^2}\right]\right.$$

$$+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial Y}\right)^2\left[(\gamma-1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma+1)\frac{\partial^2\phi_1}{\partial Y^2}\right]$$

$$+\frac{\partial\phi_2}{\partial X}\left[(\gamma+1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_1}{\partial Y^2}\right]$$

$$+\frac{\partial\phi_1}{\partial X}\left[(\gamma+1)\frac{\partial^2\phi_2}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_2}{\partial Y^2}\right]$$

$$\left.+2\left(\frac{\partial\phi_1}{\partial X}\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_1}{\partial X\partial Y}+\frac{\partial\phi_2}{\partial Y}\frac{\partial^2\phi_1}{\partial X\partial Y}+\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_2}{\partial X\partial Y}\right)\right\} \qquad (9)$$

These differential equations may be put into more familiar forms by introducing a new set of independent variables x and y, where

$$\left.\begin{aligned} x &= X \\ y &= \sqrt{1-M^2}\cdot Y \end{aligned}\right\} \qquad (10)$$

Thus, for $M < 1$, equation (7) is transformed into a Laplace equation and equations (8) and (9) into Poisson equations. The solution of equation (7) yields the well-known Prandtl-Glauert rule,

whereas the solutions of equations (8) and (9) provide higher approximations to the flow of a compressible fluid and thus will apply for larger departures from an undisturbed uniform flow.

The procedure to be followed in solving equations (7), (8), and (9) is very simple in principle. The first step is to obtain an expression for the velocity potential of the incompressible flow past the chosen boundary and to express it as a power series in the thickness coefficient t. Then the solution for the first approximation ϕ_1 to the compressible flow is easily obtained by analogy from the coefficient of the first power of t. The second and third approximations ϕ_2 and ϕ_3 are obtained by solving equations (8) and (9). The boundary conditions - that the flow is tangential to the solid boundary and that the disturbance to the main flow vanishes at infinity - are satisfied to the same power of the thickness coefficient t which is involved in the expression for the velocity potential ϕ.

Flow Past a Curved Surface

The solid boundary chosen for use in this paper is a symmetrical shape with cusps at both the leading and the trailing edges, thus insuring no stagnation points in a uniform flow parallel to the axis of symmetry. Appendix A contains the derivation of this shape and also the solution for the flow of an incompressible fluid past it. Appendixes B, C, and D contain the detailed calculations for ϕ_1, ϕ_2, and ϕ_3, respectively. The final expression for the velocity potential ϕ takes the following form:

$$
\begin{aligned}
\phi &= \cosh \xi \cos \eta + \frac{t}{4\beta}\left(3e^{-\xi} \cos \eta - e^{-3\xi} \cos 3\eta\right) \\
&\quad + t^2(F_1 \cos \eta + F_3 \cos 3\eta + F_5 \cos 5\eta) \\
&\quad + t^3\Bigg[G_1 \cos \eta + G_3 \cos 3\eta + G_5 \cos 5\eta + G_7 \cos 7\eta \\
&\quad + G_0\left(\frac{\sinh \xi \cos \eta}{\cosh 2\xi - \cos 2\eta} - e^{-\xi} \cos \eta - e^{-3\xi} \cos 3\eta - e^{-5\xi} \cos 5\eta\right. \\
&\quad \left.- e^{-7\xi} \cos 7\eta\right)\Bigg] + \ldots
\end{aligned}
\tag{11}
$$

where

ξ, η elliptic coordinates related to the Cartesian coordinates x and y by the equations

$$\left.\begin{aligned} x &= \cosh \xi \cos \eta \\ y &= \sinh \xi \sin \eta \end{aligned}\right\} \tag{12}$$

$\beta = \sqrt{1 - M^2}$

F_1, F_3, F_5 functions of ξ and of M given by equations (C-18)

G_1, G_3, G_5, G_7, G_0 functions of ξ and M given by equations (D-16), (D-17), (D-18), (D-19), and (D-22), respectively

Equation (11) represents the solution of the fundamental differential equation (1) that satisfies the boundary conditions at the surface of the body and at infinity, insofar as the terms inclusive of the third power of the thickness coefficient t are concerned. The coefficients of the various powers of t are exact and are valid for all values of the Mach number M from zero up to but not including unity. On the other hand, the method of Poggi yields the components of the fluid velocity in the form of power series in M^2, the coefficients of which are exact and valid for the entire range of values of the thickness coefficient t. Appendix E contains the solution of the problem of this paper by the method of Poggi as far as the M^2 terms are concerned.

Because the iteration process and the Poggi method yield solutions of the same equation (1) in the form of power series in t and in M^2, respectively, the two methods must agree in the range common to them; that is, the iteration expression for the fluid velocity at the solid boundary, expanded according to powers of M and with all terms containing powers of M higher than the second neglected, must agree with the corresponding Poggi result, expanded according to powers of t and with all terms containing powers of t higher than the third neglected. This calculation is shown in detail in appendix F.

Numerical Applications

Calculations are now made for the velocity distribution at the surface of a bump - that is, a member of the family of shapes derived in appendix A - for several values of the Mach number. Because terms involving powers of the thickness coefficient t higher than the third have been neglected throughout the present paper, the fluid

velocity q should be expressed in the following form:

$$q = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \ . \ . \ . \qquad (13)$$

From equations (F-6) it follows easily that

$$\left.\begin{aligned}
a_1 &= -\frac{3}{2\beta}\cos 2\alpha \\
a_2 &= \frac{3}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 + \frac{3}{8\beta^2} - \frac{9}{16} + \frac{3}{4\beta}\cos 2\alpha \\
&\quad + \left[\frac{9}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 + \frac{3}{16}\,\frac{6+10\beta-3\beta^2}{\beta^2}\right]\cos 4\alpha \\
a_3 &= -\frac{3}{16}(1-\beta) + \left(G_1 + 3G_3 + 5G_5 + 7G_7\right)_0 \\
&\quad + \left[\frac{45}{64\beta} - \frac{27}{32}(1-\beta) + 2\left(3G_3 + 5G_5 + 7G_7\right)_0\right]\cos 2\alpha \\
&\quad + \left[-\frac{9}{16}\,\frac{4+5\beta+3\beta^2}{\beta} + 2\left(5G_5 + 7G_7\right)_0\right]\cos 4\alpha \\
&\quad + \left[\frac{3}{64}\,\frac{33+82\beta+14\beta^2}{\beta} + 14\left(G_7\right)_0\right]\cos 6\alpha
\end{aligned}\right\} \qquad (14)$$

The expressions for $(G_1)_0$, $(G_3)_0$, $(G_5)_0$, and $(G_7)_0$ are given at the end of appendix F, and table IV shows the calculated values for $M = 0.50$, 0.75, 0.83, and 0.90. Table V gives the calculated values of a_1, a_2, and a_3 at various positions along the profile for $M = 0.50$, 0.75, 0.83, and 0.90. With $t = 0.10$, the expressions for the maximum fluid velocity q_{max} at the surface can be written as follows:

$$
\begin{array}{lllll}
 & t & t^2 & t^3 & \\
q_{max} = 1 + & 0.17321 + & 0.02274 + & 0.00344 & \text{for } M = 0.50 \\
q_{max} = 1 + & 0.23026 + & 0.05435 + & 0.02188 & \text{for } M = 0.75 \\
q_{max} = 1 + & 0.26893 + & 0.11014 + & 0.07240 & \text{for } M = 0.83 \\
q_{max} = 1 + & 0.34412 + & 0.19268 + & 0.43784 & \text{for } M = 0.90
\end{array}
\right\} \qquad (15)
$$

An examination of the foregoing series shows that q_{max} will probably diverge for some value of M in the neighborhood of 0.83. This value of M marks the limit of irrotational potential flow and probably indicates the first appearance of a compression shock at the solid boundary. Farther on in this section a rule will be formulated for estimating this limiting value of the stream Mach number.

The velocity distribution for a profile of thickness coefficient $t = 0.10$ is calculated by means of table V and equation (13). Table VI lists the values of q for $M = 0.50$, 0.75, and 0.83 and figure 1 shows the corresponding graphs. The broken curve represents the velocity distribution for $t = 0.10$ and $M = 0.50$ calculated according to the Poggi method. (See table III.) The curves of figure 1 show the agreement between the values of q calculated by means of the Poggi and the iteration methods for $M = 0.50$ and also the gradual change in curvature of the velocity-distribution curves in the neighborhood of the leading and the trailing edges as the stream Mach number is increased.

The critical Mach number, defined as that value of the stream Mach number at which the local fluid velocity first attains the local speed of sound, is calculated as follows:

In equation (5), $(u^2 + v^2)$ is put equal to $\frac{c^2}{U^2}$, or

$$
q_{cr}^2 = \frac{1}{\gamma + 1}\left(\frac{2}{M^2} + \gamma - 1\right) \qquad (16)
$$

Table VII lists values of q_{cr} for various values of the Mach number M. From equations (14) with $\alpha = \frac{\pi}{2}$ and the expressions for $(G_1)_0$, $(G_3)_0$, $(G_5)_0$, and $(G_7)_0$, it follows that the maximum velocity at the boundary is given by

$$q_{max} = 1 + \frac{3}{2\beta}t + \left[\frac{3}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2 + \frac{3}{8}\frac{4 + 3\beta - 3\beta^2}{\beta^2}\right]t^2$$

$$+ \left\{\frac{3}{64}\left[-16 - 16\frac{M^2}{\beta} + 12\frac{3\beta + 1}{\beta^3} + 3(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\left(\frac{1 - 8\beta^2}{\beta}\right)\right]\right.$$

$$- \frac{M^2}{\beta^2}\left[-\frac{27}{320}(\gamma+1)\frac{M^2(2 + 3\beta^2)}{\beta^3} + \gamma\beta(0.6C - 10D) + \frac{0.9}{\beta}\right.$$

$$\left.\left. + \beta(-6A+42B-5C+6D) + (\gamma+1)\frac{M^2}{\beta}(-6A+33B-2.45C+3.3D)\right]\right\}t^3 + \ldots \qquad (17)$$

Table II gives the values of A, B, C, and D obtained from equation (D-3) for various values of the Mach number. Table VII lists the values of q_{max} for $t = 0.10$ calculated by means of equation (17). Values of q_{max} calculated by the Poggi method are also given in table VII. For low Mach numbers the approximate values for q_{max} obtained by the two methods agree, but for high Mach numbers the Poggi method yields values that are too low. Figure 2 shows the graphs of q_{cr} and of q_{max} as functions of the Mach number. The intersection of the two curves gives the critical Mach number. The iteration method (solid line) yields the value $M_{cr} = 0.742$ whereas the Poggi method (broken line) yields the value $M_{cr} = 0.788$.

The value of q_{cr} for $M = 0.83$ obtained from equation (16) is 1.1731. The last column in table VIII shows the values of q/q_{cr} for the shape $t = 0.10$ with $M = 0.83$. Values of $q/q_{cr} > 1$ designate a region in which the flow is supersonic whereas values of $q/q_{cr} < 1$ characterize the subsonic region. The supersonic region of flow is symmetrical with regard to the Y axis and corresponds to a calculation given in reference 1 for the flow through a nozzle, in which a similar supersonic region of flow was found at the wall at the narrowest cross section of the tube.

In order to find the extent to which the supersonic region penetrates the flow, it is sufficient to utilize for the calculation only the terms inclusive of t^2, since the series for q converge rapidly away from the solid boundary. Thus, along the Y axis, the following expression for q is obtained from equations (F-3) and (F-4):

$$q = 1 + \frac{3t}{2\beta} e^{-2\xi} + \left\{ 12Ae^{-2\xi} - 60Be^{-4\xi} + \frac{2}{\cosh \xi} \left[C\left(4e^{-\xi} - 3e^{-3\xi} + e^{-5\xi}\right) \right.\right.$$

$$\left.\left. - D\left(8e^{-\xi} - 5e^{-3\xi} + 3e^{-5\xi}\right)\right]\right\} t^2 + \ldots \qquad (18)$$

Then for $t = 0.10$ and $M = 0.83$,

$$q = 1 + 0.40316e^{-2\xi} + 0.02753e^{-4\xi}$$

$$+ \frac{1}{\cosh \xi} \left(-0.02762e^{-\xi} + 0.01242e^{-3\xi} - 0.01520e^{-5\xi}\right)$$

The value of ξ for which $q = q_{cr} = 1.1731$ is 0.38. The corresponding value of Y obtained from equation (B-16) is

$$Y = \frac{1}{\beta} \sinh \xi$$

$$= 0.720$$

The supersonic region of flow thus extends into the fluid a distance equal to almost seven times the maximum height of the bump. By use of several chosen values of η, the constant velocity profile $q = q_{cr} = 1.1731$ can be plotted by means of equations (F-3) and (F-4). In figure 1 the lower broken curve represents this profile. The region inside the profile is completely supersonic and therefore contains real Mach lines. The region outside the profile is everywhere subsonic and therefore the Mach lines are imaginary.

The pressure coefficient $C_{p,M}$ is obtained from the expression

$$C_{p,M} = \frac{\left[1 + \frac{1}{2}(\gamma-1)M^2(1-q^2)\right]^{\gamma/\gamma-1} - 1}{\frac{1}{2}\gamma M^2} \qquad (19)$$

where

$$C_{p,M} = \frac{p - p_o}{\frac{1}{2}\rho_o U^2}$$

and q is the velocity of the compressible fluid, referred to the velocity U of the undisturbed stream.

Since, throughout this paper, terms involving powers of t higher than the third have been neglected, $C_{p,M}$ should be expressed as a power series in t. Thus, if q from equation (13) is substituted into equation (19), it follows easily that

$$C_{p,M} = -2a_1 t + \left[-\left(a_1{}^2 + 2a_2\right) + a_1{}^2 M^2\right] t^2$$

$$+ \left[-2\left(a_1 a_2 + a_3\right) + \left(a_1{}^2 + 2a_2\right) a_1 M^2 - \frac{2 - \gamma}{3} a_1{}^3 M^4\right] t^3 + \ldots \qquad (20)$$

With the help of table V, values for $C_{p,M}$ along the profile can be easily calculated for the case $t = 0.10$ and $M = 0.83$. Table IX shows these values of $C_{p,M}$ together with corresponding values calculated according to the Prandtl-Glauert rule and the von Kármán method. The Prandtl-Glauert rule is

$$C_{p,M} = \frac{C_{p,0}}{\sqrt{1 - M^2}}$$

and the relation obtained by von Kármán (reference 3) is

$$C_{p,M} = \frac{C_{p,0}}{\sqrt{1 - M^2} + \dfrac{M^2}{1 + \sqrt{1 - M^2}} \dfrac{C_{p,0}}{2}}$$

Figure 3 shows the graphs of the various calculated results. It is observed that the results of the Prandtl-Glauert and the von Kármán methods differ considerably from the results of the iteration method. The reasons for these differences are that the Prandtl-Glauert approximation, though valid for Mach numbers in the neighborhood of unity, should be utilized only for very thin shapes; whereas the

von Kármán method, though applicable to any reasonable shape, is no longer valid for Mach numbers beyond the critical value.

As final numerical applications of the results of this paper, the maximum values of the negative pressure coefficient $-\left(C_{p,M}\right)_{max}$, the critical pressure coefficient $-\left(C_{p,M}\right)_{cr}$, and the limiting pressure coefficient $-\left(C_{p,M}\right)_{lim}$ are calculated for various values of the thickness coefficient and of the stream Mach number.

The maximum values of the negative pressure coefficient for various values of t and of M are obtained by means of equations (17) and (20). Table X shows values of a_1, a_2, and a_3 defined, respectively, as the coefficients of t, t^2, and t^3 in equation (17). The corresponding values of $C_{p,M}$, obtained by means of equation (20), are listed in table XI, together with values of $C_{p,M}$ calculated by the von Kármán method. Figure 4 shows the variation of $\left(C_{p,M}\right)_{max}$ with Mach number for several values of t.

The critical pressure coefficient $\left(C_{p,M}\right)_{cr}$ is calculated by means of the following expression obtained by substituting for q^2 in equation (19) the expression for q_{cr}^2 from equation (16):

$$\left(C_{p,M}\right)_{cr} = \frac{2}{\gamma M^2}\left\{-1 + \left[\frac{2 + (\gamma-1)M^2}{\gamma + 1}\right]^{\gamma/\gamma-1}\right\} \tag{21}$$

Table XII gives the values of $\left(C_{p,M}\right)_{cr}$ calculated by means of this equation, and figure 4 shows the corresponding graph. The intersections of the $\left(C_{p,M}\right)_{max}$ curves with the $\left(C_{p,M}\right)_{cr}$ curve yield the critical values of the stream Mach number which are listed in table XII.

As noted once before, for a given value of t, the series for q (equation (13)) apparently diverges for a definite value of the stream Mach number. It is reasonable to assume that this value of the Mach number marks the limit of irrotational potential flow and also probably indicates the first appearance of a compression shock at the solid boundary. Equation (17) for q_{max} can be used to estimate the limiting values of M according to the following rough criterion. By means of table X, expressions for q_{max}, in the

form of power series in t, can be obtained for any value of the stream Mach number in the range $0 \leq M < 1$. For a given value of the thickness coefficient t, a series for q_{max} can then be found so that a term-for-term comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ yields a decreasing sequence for the ratio of corresponding terms. The value of the Mach number thus obtained is chosen as the limiting value of the Mach number. Table XII lists both the values of $\left(C_{p,M}\right)_{lim}$ and the corresponding values of M for different values of t. Figure 4 shows the curve connecting the limiting values of $C_{p,M}$. The region between this curve and the $\left(C_{p,M}\right)_{cr}$ curve represents the supersonic range without compression shocks. It is emphasized that the mathematical procedure outlined in this paragraph is a highly speculative one.

The maximum speed that can be attained by an adiabatic fluid is equal to the speed when $p = \rho = c = 0$ and is given by Bernoulli's equation

$$q_{abs}^2 = 1 + \frac{2}{(\gamma-1)M^2}$$

Values of the pressure coefficient

$$\left(C_{p,M}\right)_{abs} = -\frac{2}{\gamma M^2} \tag{22}$$

are listed in table XIII for various values of the undisturbed-stream Mach number M, and figure 4 shows the corresponding curve. The region between this curve and the $\left(C_{p,M}\right)_{lim}$ curve represents the supersonic range with compression shock. On and beyond the $\left(C_{p,M}\right)_{abs}$ curve the adiabatic fluid ceases to exist; that is, absolute vacuum prevails.

In conclusion, it may be remarked that the results of von Kármán, shown by the small circles in figure 4, are obtained independently of any assumption concerning the shape of the solid boundary, whereas the results of this paper were obtained for a specific family of shapes. As shown by the curves of figure 4, nevertheless, the results of this paper agree with those obtained by von Kármán's method. This agreement has some justification, for the values of $\left(C_{p,M}\right)_{max}$ and M_{cr} depend mainly on the dimensions of a body - that is,

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thickness coefficient - and not on its shape. The hodograph method as employed by von Kármán, however, yields results that cease to be valid when the local Mach number equals or exceeds unity, whereas the iteration method utilized in the present paper is valid for local Mach numbers greater than unity and, for the family of shapes chosen, yields some information with regard to a supersonic region of flow without shock. The limiting value of M for a given shape, beyond which supersonic flow without shock does not exist, appears to depend on the convergence of the power series in t for the velocity. Although only a few terms of the series have been obtained and therefore the limiting value of M cannot be given precisely, nevertheless it is believed that a reasonable estimate of the value for M_{lim} can be made by the comparison test with the harmonic series.

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APPENDIX A

THE INCOMPRESSIBLE FLOW PAST A CURVED SURFACE

In the search for a shape which satisfies the conditions that it be thin and that it possess no stagnation points, the first thought is of a straight-line segment. It is well known that a straight-line segment of length $4c$ is obtained from a circle of radius c by means of a Joukowski transformation. If Z denotes the plane of the segment and Z' the plane of the circle, then

$$Z = Z' + \frac{c^2}{Z'} \tag{A-1}$$

The singular points of this transformation are determined by the equation

$$\frac{dZ}{dZ'} = \left(1 + \frac{c}{Z'}\right)\left(1 - \frac{c}{Z'}\right) \tag{A-2}$$

which shows zeros at $Z' = \pm c$. In order to raise the top surface and lower the bottom surface of the line segment, it is necessary only to place two additional zeros at $Z' = \pm id$ where $d < c$. Analogous to equation (A-2),

$$\frac{dZ}{dZ'} = \left(1 + \frac{c}{Z'}\right)\left(1 - \frac{c}{Z'}\right)\left(1 + \frac{id}{Z'}\right)\left(1 - \frac{id}{Z'}\right) \tag{A-3}$$

Then, on integration of equation (A-3), it follows that

$$Z = Z' + \frac{c^2 - d^2}{Z'} + \frac{c^2d^2}{3Z'^3} \tag{A-4}$$

The parametric equations of the shape in the Z plane corresponding to the circle of radius c in the Z' plane are obtained by substituting $Z' = ce^{i\theta}$ in equation (A-4); thus,

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$$\left.\begin{aligned} X &= 2c \cos\theta - \frac{d^2}{3c}(3\cos\theta - \cos 3\theta) \\ Y &= \frac{d^2}{3c}(3\sin\theta - \sin 3\theta) \end{aligned}\right\} \tag{A-5}$$

The family of shapes given by equations (A-5) includes, on one hand, the straight-line segment with $d = 0$ and, on the other hand, a shape having four cusps symmetrically placed with respect to the coordinate axes with $d = c$. For $0 < d < c$, the slope dY/dX is zero for $\theta = 0$, $\pi/2$, and π. The shape thus has cusps at $X = \pm 2c\left(1 - \frac{d^2}{3c^2}\right)$, $Y = 0$, and the maximum and minimum points are at $X = 0$, $Y = \frac{4}{3}\frac{d^2}{c}$, and $X = 0$, $Y = -\frac{4}{3}\frac{d^2}{c}$, respectively.

The complex potential for a circular cylinder of radius c, fixed in a stream of uniform velocity U in the positive direction of the real axis, is given by

$$F = U\left(Z' + \frac{c^2}{Z'}\right) \tag{A-6}$$

The complex velocity past the corresponding shape in the Z plane is

$$u - iv = \frac{dF}{dZ} = \frac{dF}{dZ'}\frac{dZ'}{dZ}$$

or

$$q^2 = u^2 + v^2 = \frac{dF}{dZ'}\frac{d\overline{F}}{d\overline{Z}'}\frac{dZ'}{dZ}\frac{d\overline{Z}'}{d\overline{Z}}$$

By means of equations (A-4) and (A-6), it follows that

$$q^2 = U^2 \frac{\left(1 - \frac{c^2}{Z'^2}\right)\left(1 - \frac{c^2}{\overline{Z}'^2}\right)}{\left(1 - \frac{c^2 - d^2}{Z'^2} - \frac{c^2d^2}{Z'^4}\right)\left(1 - \frac{c^2 - d^2}{\overline{Z}'^2} - \frac{c^2d^2}{\overline{Z}'^4}\right)}$$

and at the surface of the profile, where $Z' = ce^{i\theta}$,

$$q^2 = \frac{U^2}{1 + 2\frac{d^2}{c^2}\cos 2\theta + \left(\frac{d^2}{c^2}\right)^2} \tag{A-7}$$

It will be convenient to consider F, Z, and Z' as nondimensional quantities. Thus, in the plane Z' the unit of length is the radius c of the circle; whereas, in the plane Z, the unit of length is the semichord s of the shape. Then Z', Z, and F denote Z'/c, Z/s, and F/Us, respectively. Also, the ratio d^2/c^2 is designated by ϵ.

According to equations (A-5), the semichord s is given by

$$s = 2c\left(1 - \frac{1}{3}\epsilon\right)$$

and the thickness coefficient t by

$$t = \frac{2\epsilon}{3 - \epsilon}$$

With the introduction of these new designations, equations (A-4), (A-5), (A-6), and (A-7) become, respectively,

$$Z = \frac{2 + t}{4} Z' + \frac{1 - t}{2}\frac{1}{Z'} + \frac{t}{4}\frac{1}{Z'^3} \tag{A-8}$$

$$\left.\begin{aligned} X &= \frac{2+t}{4}\cos\theta - \frac{t}{4}(3\cos\theta - \cos 3\theta) \\ Y &= \frac{t}{2}(3\sin\theta - \sin 3\theta) \end{aligned}\right\} \tag{A-9}$$

$$F = \frac{2+t}{4}\left(Z' + \frac{1}{Z'}\right) \tag{A-10}$$

$$q^2 = \frac{1}{1 + 2\epsilon\cos 2\theta + \epsilon^2} \tag{A-11}$$

As a numerical example of the use of equations (A-9), (A-10), and (A-11), table I gives the coordinates for the shape $t = 0.10$ together with the velocity and pressure distributions along the profile. The pressure coefficient C_p is calculated by means of Bernoulli's equation:

$$C_p = \frac{p - p_o}{\frac{1}{2}\rho U^2} = 1 - q^2$$

where

p local static pressure

p_o static pressure in undisturbed stream

ρ density of fluid

Figure 1 shows the curves of the shape and the velocity distribution, and figure 3 shows the graph of the pressure distribution. Because the body is placed symmetrically with respect to the undisturbed stream, the flow is identical with that over a solid boundary composed of the X axis from $X = \infty$ to $X = 1$, the upper surface of the shape with $0 \leqq \theta \leqq \pi$, and the X axis from $X = -1$ to $X = -\infty$. This boundary is called a bump.

APPENDIX B

THE COMPRESSIBLE FLOW PAST A BUMP

Before proceeding with the iteration process, developments for ϕ and Y in positive, integral powers of the thickness parameter t must be obtained. From equation (A-8),

$$t = 4\,\frac{Z - \frac{1}{2}\left(Z' + \frac{1}{Z'}\right)}{Z'\left(1 - \frac{1}{Z'^2}\right)^2} \tag{B-1}$$

and from equation (A-10),

$$F = \left[\frac{1}{2} + \frac{Z - \frac{1}{2}\left(Z' + \frac{1}{Z'}\right)}{Z'\left(1 - \frac{1}{Z'^2}\right)^2}\right]\left(Z' + \frac{1}{Z'}\right) \tag{B-2}$$

By means of a Taylor expansion in the neighborhood of $t = 0$, it is possible to express F as a power series in t in which the coefficients are functions of Z. Thus, according to equation (B-1), $t = 0$ when

$$Z = \frac{1}{2}\left(Z' + \frac{1}{Z'}\right)$$

or

$$Z' = Z + \sqrt{Z^2 - 1}$$

where the positive sign is taken with the radical because the points at infinity of the Z and Z' planes must correspond. Now

$$F = (F)_{t=0} + t\left(\frac{dF}{dt}\right)_{t=0} + \ldots$$

If, on the right-hand sides of equations (B-1) and (B-2), Z is considered constant and Z' is considered the parameter, it follows that

$$F = Z + \left[\frac{1}{2} Z - \left(Z^2 - 1\right)\left(Z - \sqrt{Z^2 - 1}\right)\right] t + \ldots$$

Since $F = \phi + i\psi$, where ψ is the stream function,

$$\phi = X + \frac{1}{2}\left[X - \left(Z^2 - 1\right)\left(Z - \sqrt{Z^2 - 1}\right) - \left(\bar{Z}^2 - 1\right)\left(\bar{Z} - \sqrt{\bar{Z}^2 - 1}\right)\right] t + \ldots \qquad \text{(B-3)}$$

In a similar manner, equations (A-9) can be written as follows:

$$\left.\begin{aligned} t &= -\frac{X - \cos\theta}{\sin^2\theta \cos\theta} \\ Y &= -(X - \cos\theta)\tan\theta \end{aligned}\right\} \qquad \text{(B-4)}$$

If X is considered constant and θ is considered the parameter, a Taylor expansion yields

$$Y = \left(1 - X^2\right)^{3/2}\left[t - 3X^2t^2 + \frac{3}{2}\left(8X^4 - 3X^2\right)t^3 - \ldots\right] \qquad \text{(B-5)}$$

Determination of ϕ_1

By means of the transformation $x = X$, $y = \beta Y$, where $\beta = \sqrt{1 - M^2}$, equation (7) for ϕ_1 becomes

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \qquad \text{(B-6)}$$

By analogy with the coefficient of t in equation (B-3) for ϕ, it is suggested that

$$\phi_1 = k\left[x - \left(z^2 - 1\right)\left(z - \sqrt{z^2 - 1}\right) - \left(\bar{z}^2 - 1\right)\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)\right] \tag{B-7}$$

where the coefficient k is determined by means of the boundary condition

$$\frac{\partial \phi}{\partial X}\frac{dY}{dX} = \frac{\partial \phi}{\partial Y}$$

or

$$\frac{\partial \phi}{\partial x}\frac{dy}{dx} = \beta^2 \frac{\partial \phi}{\partial y} \tag{B-8}$$

and where the boundary, obtained from equation (B-5), is given by

$$y = \beta\left(1 - x^2\right)^{3/2}\left[t - 3x^2t^2 + \frac{3}{2}\left(8x^4 - 3x^2\right)t^3 \quad . \; . \; .\right] \tag{B-9}$$

It is clear that the boundary condition need be satisfied only to the same degree in t as is involved in the development of the velocity potential ϕ. Thus,

$$\phi = x + k\left[x - \left(z^2 - 1\right)\left(z - \sqrt{z^2 - 1}\right) - \left(\bar{z}^2 - 1\right)\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)\right]t + \; . \; . \; . \; . \tag{B-10}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} = 1 + k\Big[1 - 2z\left(z - \sqrt{z^2 - 1}\right) - 2\bar{z}\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)$$

$$+ \sqrt{z^2 - 1}\left(z - \sqrt{z^2 - 1}\right) + \sqrt{\bar{z}^2 - 1}\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)\Big]t + \; . \; . \; . \tag{B-11}$$

$$\frac{\partial \phi}{\partial y} = i\left(\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}}\right) = -ik\Big[2z\left(z - \sqrt{z^2 - 1}\right) - 2\bar{z}\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)$$

$$- \sqrt{z^2 - 1}\left(z - \sqrt{z^2 - 1}\right) + \sqrt{\bar{z}^2 - 1}\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)\Big]t + \; . \; . \; . \tag{B-12}$$

$$\frac{dy}{dx} = -3\beta x\sqrt{1 - x^2}\; t + \; . \; . \; . \tag{B-13}$$

Suppose that $x = \cos\alpha$. Then, insofar as the terms of the first power in t are concerned, it follows that, on the boundary,

$$z = \cos\alpha + i\beta t \sin^3\alpha$$

$$\bar{z} = \cos\alpha - i\beta t \sin^3\alpha$$

$$\sqrt{z^2 - 1} = i \sin\alpha \left(1 - i\beta t \sin^3\alpha\right)$$

$$\sqrt{\bar{z}^2 - 1} = -i \sin\alpha \left(1 + i\beta t \sin^3\alpha\right)$$

Then, on the boundary,

$$\frac{\partial \phi}{\partial x} = 1 - 3kt \sin 2\alpha$$

$$\frac{\partial \phi}{\partial y} = -3kt \sin 2\alpha$$

$$\frac{dy}{dx} = -\frac{3}{2}\beta t \sin 2\alpha$$

The boundary condition, equation (B-8), then yields

$$k = \frac{1}{2\beta}$$

Therefore

$$\phi = x + \frac{1}{2\beta}\left[x - \left(z^2 - 1\right)\left(z - \sqrt{z^2 - 1}\right) - \left(\bar{z}^2 - 1\right)\left(\bar{z} - \sqrt{\bar{z}^2 - 1}\right)\right] t + \ldots \tag{B-14}$$

This expression for ϕ can be put into a simple form by means of the transformation

$$z = \cosh \zeta \tag{B-15}$$

where

$$\zeta = \xi + i\eta$$

Also,

$$\left.\begin{aligned} x &= \cosh\xi \, \cos\eta \\ y &= \sinh\xi \, \sin\eta \end{aligned}\right\} \qquad \text{(B-16)}$$

Then from equation (B-14) it follows that

$$\left.\begin{aligned} \phi_0 &= \cosh\xi \, \cos\eta \\ \phi_1 &= \frac{1}{4\beta}\left(3e^{-\xi}\cos\eta - e^{-3\xi}\cos 3\eta\right) \end{aligned}\right\} \qquad \text{(B-17)}$$

Inversion of Equations (B-16)

The relationship between the rectangular Cartesian coordinates X, Y and the elliptic coordinates ξ, η is obtained as follows:

Invert equations (B-16) and solve for ξ and η; thus

$$\left(\frac{x}{\cosh\xi}\right)^2 + \left(\frac{y}{\sinh\xi}\right)^2 = 1$$

and

$$\left(\frac{x}{\cosh\xi}\right)^2 - \left(\frac{y}{\sinh\xi}\right)^2 = 1$$

Solve for $\sinh^2\xi$,

$$2\sinh^2\xi = -b + \sqrt{b^2 + 4y^2}$$

and solve for $\sin^2\eta$

$$2\sin^2\eta = b + \sqrt{b^2 + 4y^2}$$

where

$$b = 1 - \left(x^2 + y^2\right)$$

By means of the transformation

$$x = X$$

$$y = \beta Y$$

it follows that

$$\left.\begin{aligned} 2\sinh^2\xi &= -b + \sqrt{b^2 + 4\beta^2 Y^2} \\ 2\sin^2\eta &= b + \sqrt{b^2 + 4\beta^2 Y^2} \end{aligned}\right\} \tag{B-18}$$

where

$$b = 1 - \left(X^2 + \beta^2 Y^2\right)$$

APPENDIX C

DETERMINATION OF ϕ_2

The differential equation (8) for ϕ_2 in terms of the coordinates x and y becomes

$$\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2} = \frac{1-\beta^2}{\beta^2}\left[(\gamma+1)\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial x^2}\right.$$

$$\left. + (\gamma-1)\beta^2\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial y^2} + 2\beta^2\frac{\partial\phi_1}{\partial y}\frac{\partial^2\phi_1}{\partial x\partial y}\right] \tag{C-1}$$

By means of the symbolic relations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z\partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$\frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z\partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial x\partial y} = i\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2}\right)$$

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and the transformation

$$z = \cosh \zeta$$

equation (C-1) can be expressed in terms of the complex variables ζ and $\bar{\zeta}$ as follows:

$$\frac{4}{\sinh \zeta \sinh \bar{\zeta}} \frac{\partial^2 \phi_2}{\partial \zeta \partial \bar{\zeta}} = \frac{1 - \beta^2}{\beta^2} \left\{ \left[(\gamma+1) - (\gamma-1)\beta^2\right] \left(\frac{1}{\sinh \zeta} \frac{\partial \phi_1}{\partial \zeta} \right. \right.$$

$$\left. + \frac{1}{\sinh \bar{\zeta}} \frac{\partial \phi_1}{\partial \bar{\zeta}} \right) \left(\frac{1}{\sinh^2 \zeta} \frac{\partial^2 \phi_1}{\partial \zeta^2} - \frac{\cosh \zeta}{\sinh^3 \zeta} \frac{\partial \phi_1}{\partial \zeta} + \frac{1}{\sinh^2 \bar{\zeta}} \frac{\partial^2 \phi_1}{\partial \bar{\zeta}^2} - \frac{\cosh \bar{\zeta}}{\sinh^3 \bar{\zeta}} \frac{\partial \phi_1}{\partial \bar{\zeta}} \right)$$

$$- 2\beta^2 \left(\frac{1}{\sinh \zeta} \frac{\partial \phi_1}{\partial \zeta} - \frac{1}{\sinh \bar{\zeta}} \frac{\partial \phi_1}{\partial \bar{\zeta}} \right) \left(\frac{1}{\sinh^2 \zeta} \frac{\partial^2 \phi_1}{\partial \zeta^2} \right.$$

$$\left. \left. - \frac{\cosh \zeta}{\sinh^3 \zeta} \frac{\partial \phi_1}{\partial \bar{\zeta}^2} - \frac{1}{\sinh^2 \bar{\zeta}} \frac{\partial^2 \phi_1}{\partial \bar{\zeta}^2} + \frac{\cosh \bar{\zeta}}{\sinh^3 \bar{\zeta}} \frac{\partial \phi_1}{\partial \bar{\zeta}} \right) \right\} \tag{C-2}$$

Now, equation (B-17) for ϕ_1 can be rewritten as follows:

$$\phi_1 = \frac{1}{8\beta} \left(3e^{-\zeta} - e^{-3\zeta} + 3e^{-\bar{\zeta}} - e^{-3\bar{\zeta}} \right) \tag{C-3}$$

The substitution of this expression for ϕ_1 into the right-hand side of equation (C-2) then yields the following differential equation for ϕ_2:

$$\frac{\partial^2 \phi_2}{\partial \xi^2} + \frac{\partial^2 \phi_2}{\partial \eta^2} = \frac{9}{8} \frac{1 - \beta^2}{\beta^4} \left\{ \left[(\gamma+1) + (3-\gamma)\beta^2\right] \left(e^{-5\xi} - e^{-3\xi} \right) \cos \eta \right.$$

$$\left. + (\gamma-1)\left(1 - \beta^2\right)\left(e^{-5\xi} \cos 3\eta - e^{-3\xi} \cos 5\eta\right) \right\} \tag{C-4}$$

The right-hand side of this equation suggests a solution of the form

$$\phi_2 = F_1(\xi)\cos\eta + F_3(\xi)\cos 3\eta + F_5(\xi)\cos 5\eta \tag{C-5}$$

If this expression for ϕ_2 is inserted into equation (C-4) and the coefficients of $\cos\eta$, $\cos 3\eta$, and $\cos 5\eta$ are equated on both sides of the equation, the following differential equations for $F_1(\xi)$, $F_3(\xi)$, and $F_5(\xi)$ are obtained:

$$\frac{d^2F_1}{d\xi^2} - F_1 = \frac{9}{8}\,\frac{1-\beta^2}{\beta^4}\left[(\gamma+1) + (3-\gamma)\beta^2\right]\left(e^{-5\xi} - e^{-3\xi}\right) \tag{C-6}$$

$$\frac{d^2F_3}{d\xi^2} - 9F_3 = \frac{9}{8}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 e^{-5\xi} \tag{C-7}$$

$$\frac{d^2F_5}{d\xi^2} - 25F_5 = -\frac{9}{8}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 e^{-3\xi} \tag{C-8}$$

The solutions for F_1, F_3, and F_5 are easily obtained and are as follows:

$$F_1 = \frac{3}{64}\,\frac{1-\beta^2}{\beta^4}\left\{A_1e^{-\xi} + \left[(\gamma+1) + (3-\gamma)\beta^2\right]\left(e^{-5\xi} - 3e^{-3\xi}\right)\right\} \tag{C-9}$$

$$F_3 = \frac{9}{128}\,\frac{1-\beta^2}{\beta^4}\left[A_3e^{-3\xi} + (\gamma+1)\left(1-\beta^2\right)e^{-5\xi}\right] \tag{C-10}$$

$$F_5 = \frac{9}{128}\,\frac{1-\beta^2}{\beta^4}\left[A_5e^{-5\xi} + (\gamma+1)\left(1-\beta\right)e^{-3\xi}\right] \tag{C-11}$$

where A_1, A_3, and A_5 are arbitrary constants. It is noted that, in general, the expressions for F_1, F_3, and F_5 should each contain two arbitrary constants; however, the condition that F_1, F_3, and F_5 vanish at infinity requires that the omitted arbitrary constants be taken equal to zero.

The arbitrary constants A_1, A_3, and A_5 are determined by means of the boundary equation (B-8). In terms of the variables ξ and η equation (B-8) takes the form

$$\left(\sinh \xi \cos \eta \frac{\partial \phi}{\partial \xi} - \cosh \xi \sin \eta \frac{\partial \phi}{\partial \eta}\right)\frac{dy}{dx} =$$

$$\beta^2 \left(\cosh \xi \sin \eta \frac{\partial \phi}{\partial \xi} + \sinh \xi \cos \eta \frac{\partial \phi}{\partial \eta}\right) \qquad \text{(C-12)}$$

and the velocity potential ϕ has the form

$$\phi = \cosh \xi \cos \eta + \frac{t}{4\beta}\left(3e^{-\xi} \cos \eta - e^{-3\xi} \cos 3\eta\right)$$

$$+ t^2 \left(F_1 \cos \eta + F_3 \cos 3\eta + F_5 \cos 5\eta\right) + \dots \qquad \text{(C-13)}$$

With $x = \cos \alpha$, the boundary is given by

$$y = \beta t \sin^3\alpha - 3\beta t^2 \sin^3\alpha \cos^2\alpha + \frac{3}{2}\beta t^3 \left(8\sin^3\alpha \cos^4\alpha - 3\sin^3\alpha \cos^2\alpha\right) - \dots$$

and

$$\frac{dy}{dx} = -\frac{3}{2}\beta t \sin 2\alpha + \frac{3}{8}\beta t^2 (2\sin 2\alpha + 5\sin 4\alpha)$$

$$- \frac{3}{16}\beta t^3 (9\sin 4\alpha + 14\sin 6\alpha) + \dots \qquad \text{(C-14)}$$

At the boundary, inclusive of terms containing the third power of t,

$$\left.\begin{aligned}
p &= 1 - \left(x^2 + y^2\right) = \sin^2\alpha\left(1 - \beta^2 t^2 \sin^4\alpha + 6\beta^2 t^3 \sin^4\alpha\cos^2\alpha\right) \\
\sin\xi &= \beta t \sin^2\alpha\left(1 - 3t\cos^2\alpha\right) \\
\cosh\xi &= 1 + \frac{1}{2}\beta^2 t^2 \sin^4\alpha - 3\beta^2 t^3 \sin^4\alpha\cos^2\alpha \\
e^{-\xi} &= 1 - \beta t \sin^2\alpha + \frac{1}{2}\beta^2 t^2 \sin^4\alpha + 3\beta t^2 \sin^2\alpha\cos^2\alpha \\
\sin\eta &= \sin\alpha\left(1 - \frac{1}{2}\beta^2 t^2 \sin^4\alpha + \frac{1}{2}\beta^2 t^2 \sin^2\alpha\right. \\
&\qquad \left. + 3\beta^2 t^3 \sin^4\alpha\cos^2\alpha - 3\beta^2 t^3 \sin^2\alpha\cos^2\alpha\right) \\
\cos\eta &= \cos\alpha\left(1 - \frac{1}{2}\beta^2 t^2 \sin^4\alpha + 3\beta^2 t^3 \sin^4\alpha\cos^2\alpha\right)
\end{aligned}\right\} \qquad \text{(C-15)}$$

At the boundary then,

$$\left.\begin{aligned}
\frac{dF_1}{d\xi} &= \frac{3}{64}\,\frac{1-\beta^2}{\beta^4}\left\{-A_1 + 4\left[(\gamma+1) + (3-\gamma)\beta^2\right]\right\} \\
\frac{dF_3}{d\xi} &= \frac{9}{128}\,\frac{1-\beta^2}{\beta^4}\left[-3A_3 - 5(\gamma+1)\left(1-\beta^2\right)\right] \\
\frac{dF_5}{d\xi} &= \frac{9}{128}\,\frac{1-\beta^2}{\beta^4}\left[-5A_5 - 3(\gamma+1)\left(1-\beta^2\right)\right]
\end{aligned}\right\} \qquad \text{(C-16)}$$

Hence, with all terms involving powers of t higher than the second excluded, the boundary condition (equation (C-12)) yields the

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following expressions for the arbitrary constants A_1, A_3, and A_5:

$$
\left.
\begin{aligned}
A_1 &= 4\left[(\gamma+1) + (3-\gamma)\beta^2\right] - 8\,\frac{\beta^3(1-2\beta)}{1-\beta^2} \\
A_3 &= -\frac{5}{3}(\gamma+1)\left(1-\beta^2\right) - \frac{8}{3}\,\frac{\beta^2\left(1+\beta+2\beta^2\right)}{1-\beta^2} \\
A_5 &= -\frac{3}{5}(\gamma+1)\left(1-\beta^2\right) + \frac{8}{15}\,\frac{\beta^2\left(3+5\beta+2\beta^2\right)}{1-\beta^2}
\end{aligned}
\right\} \qquad \text{(C-17)}
$$

The expressions for F_1, F_3, and F_5 then become:

$$
\left.
\begin{aligned}
F_1(\xi) &= \frac{3}{64}\,\frac{1-\beta^2}{\beta^4}\left[(\gamma+1) + (3-\gamma)\beta^2\right]\left(4e^{-\xi} - 3e^{-3\xi} + e^{-5\xi}\right) \\
&\quad - \frac{3}{8}\,\frac{1-2\beta}{\beta}\,e^{-\xi} \\
F_3(\xi) &= \frac{3}{128}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\left(3e^{-5\xi} - 5e^{-3\xi}\right) \\
&\quad - \frac{3}{16}\,\frac{1+\beta+2\beta^2}{\beta^2}\,e^{-3\xi} \\
F_5(\xi) &= \frac{9}{640}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\left(5e^{-3\xi} - 3e^{-5\xi}\right) \\
&\quad + \frac{3}{80}\,\frac{3+5\beta+2\beta^2}{\beta^2}\,e^{-5\xi}
\end{aligned}
\right\} \qquad \text{(C-18)}
$$

Equation (C-13), together with equations (C-18), represents a solution of the fundamental differential equation (1) and satisfies the boundary conditions at the surface of the solid boundary and at infinity, insofar as the terms inclusive of the second power in t are concerned. The present process can be extended to the higher terms in the development of the velocity potential ϕ, but it can readily be seen from the complexity of the right-hand side of equation (9) that the labor involved would increase rapidly with the degree of approximation. In view of the importance of the problem, it has nevertheless been thought worth while to extend the calculation to the third approximation ϕ_3.

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APPENDIX D

DETERMINATION OF ϕ_3

The differential equation (9) for ϕ_3 in terms of the complex variables z and $\bar{z}$ can be written as follows:

$$4\phi_{3z\bar{z}} = \frac{1-\beta^2}{\beta^2}\left\{\frac{1}{2}\left[(\gamma+1)\left(1-\beta^2\right)^2 + 4\beta^2\right]\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\left(\phi_{1z}{}^2 + \phi_{1\bar{z}}{}^2\right)\right.$$

$$+ (\gamma+1)\left(1-\beta^4\right)\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\phi_{1z}\phi_{1\bar{z}}$$

$$+ \left[(\gamma+1)\left(1-\beta^2\right)+2\beta^2\right]\left[\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\left(\phi_{2z} + \phi_{2\bar{z}}\right) + \left(\phi_{1z} + \phi_{1\bar{z}}\right)\left(\phi_{2zz} + \phi_{2\bar{z}\bar{z}}\right)\right]$$

$$+ 2\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]\left(\phi_{1z} + \phi_{1\bar{z}}\right)\phi_{2z\bar{z}}$$

$$- 2\beta^2\left[\left(\phi_{1z}{}^2 - \phi_{1\bar{z}}{}^2\right)\left(\phi_{1zz} - \phi_{1\bar{z}\bar{z}}\right) + \left(\phi_{2z} - \phi_{2\bar{z}}\right)\left(\phi_{1zz} - \phi_{1\bar{z}\bar{z}}\right)\right.$$

$$\left.\left. + \left(\phi_{1z} - \phi_{1\bar{z}}\right)\left(\phi_{2zz} - \phi_{2\bar{z}\bar{z}}\right)\right]\right\} \tag{D-1}$$

where, for example,

$$\phi_{3z\bar{z}} = \frac{\partial^2\phi_3}{\partial z\partial\bar{z}}$$

Again, complex variables ζ and $\bar{\zeta}$ are introduced in place of z and $\bar{z}$ by means of the transformation

$$z = \cosh\,\zeta$$

According to equation (C-3),

$$\phi_1 = \frac{1}{8\beta}\left(3e^{-\zeta} - e^{-3\zeta} + 3e^{-\bar{\zeta}} - e^{-3\bar{\zeta}}\right) \qquad \text{(C-3)}$$

Consider the term

$$T_1 = \frac{1}{2}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\left(\phi_{1z}{}^2 + \phi_{1\bar{z}}{}^2\right)$$

$$+ (\gamma+1)\left(1-\beta^4\right)\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\phi_{1z}\phi_{1\bar{z}} - 2\beta^2\left(\phi_{1zz} - \phi_{1\bar{z}\bar{z}}\right)\left(\phi_{1z}{}^2 - \phi_{1\bar{z}}{}^2\right)$$

Now

$$\phi_{1z} = -\frac{3}{4\beta}\, e^{-2\zeta}$$

$$\phi_{1zz} = \frac{3}{2\beta}\,\frac{e^{-2\zeta}}{\sinh\bar{\zeta}}$$

and

$$\phi_{1z}{}^2 + \phi_{1\bar{z}}{}^2 = \frac{9}{8\beta^2}\, e^{-4\xi}\cos 4\eta .$$

$$\phi_{1z}{}^2 - \phi_{1\bar{z}}{}^2 = -\frac{9i}{8\beta^2}\, e^{-4\xi}\sin 4\eta$$

$$\phi_{1z}\phi_{1\bar{z}} = \frac{9}{16\beta^2}\, e^{-4\xi}$$

$$\phi_{1zz} + \phi_{1\bar{z}\bar{z}} = \frac{3}{2\beta}\,\frac{e^{-\xi}\cos 3\eta - e^{-3\xi}\cos\eta}{\sinh\zeta\,\sinh\bar{\zeta}}$$

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$$\phi_{1zz} - \phi_{1\bar{z}\bar{z}} = -\frac{3i}{2\beta}\,\frac{e^{-\xi}\sin 3\eta - e^{-3\xi}\sin\eta}{\sinh\zeta\,\sinh\bar{\zeta}}$$

It follows then that

$T_1 \sinh\zeta \sinh\bar{\zeta} =$

$$\frac{27}{64\beta^3}\left(\left\{\left[(\gamma+1)\left(1-\beta^2\right)^2 + 8\beta^2\right]e^{-5\xi} - 2(\gamma+1)\left(1-\beta^4\right)e^{-7\xi}\right\}\cos\eta\right.$$

$$+\left\{2(\gamma+1)\left(1-\beta^4\right)e^{-5\xi} - \left[(\gamma+1)\left(1-\beta^2\right) + 8\beta^2\right]e^{-7\xi}\right\}\cos 3\eta$$

$$\left.-\,(\gamma+1)\left(1-\beta^2\right)^2 e^{-7\xi}\cos 5\eta + (\gamma+1)\left(1-\beta^2\right)^2 e^{-5\xi}\cos 7\eta\right) \qquad \text{(D-2)}$$

In the handling of the terms on the right-hand side of equation (D-1) that contain the derivatives of ϕ_2, it is convenient to separate ϕ_2 into two parts: namely, ϕ_2', which is a function of ζ plus a function of $\bar{\zeta}$, and ϕ_2'', which is a function of ζ and $\bar{\zeta}$. Thus, if the variables ζ and $\bar{\zeta}$ are introduced into the expression for ϕ_2, it follows easily that

$$\left.\begin{aligned}
\phi_2' &= A\left(3e^{-\zeta} - e^{-3\zeta} + 3e^{-\bar{\zeta}} - e^{-3\bar{\zeta}}\right) + B\left(5e^{-3\zeta} - 3e^{-5\zeta} + 5e^{-3\bar{\zeta}} - 3e^{-5\bar{\zeta}}\right) \\
\phi_2'' &= C\left(4e^{-\zeta} - 3e^{-\zeta-2\bar{\zeta}} + e^{-2\zeta-3\bar{\zeta}} + 4e^{-\bar{\zeta}} - 3e^{-2\zeta-\bar{\zeta}} + e^{-3\zeta-2\bar{\zeta}}\right) \\
&\quad + D\left(-8e^{-\zeta} + e^{-\zeta-4\bar{\zeta}} + e^{\zeta-4\bar{\zeta}} - 8e^{-\bar{\zeta}} + e^{-4\zeta-\bar{\zeta}} + e^{-4\zeta+\bar{\zeta}}\right)
\end{aligned}\right\} \qquad \text{(D-3)}$$

where

$$A = \frac{3}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 - \frac{1-2\beta}{16\beta}$$

$$B = \frac{9}{1280}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 - \frac{3+5\beta+2\beta^2}{160\beta^2}$$

$$C = \frac{3}{128}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 + \frac{3}{32}\,\frac{1-\beta^2}{\beta^2}$$

$$D = \frac{9}{256}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2$$

Table II gives values of A, B, C, and D for values of the Mach number M ranging from zero to nearly unity. The value of γ taken is 1.405.

Consider the term

$$T_2 = \left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\left[\left(\phi_{1zz} + \phi_{1\bar{z}\bar{z}}\right)\left(\phi_2{}'_z + \phi_2{}'_{\bar{z}}\right)\right.$$

$$\left. + \left(\phi_{1z} + \phi_{1\bar{z}}\right)\left(\phi_2{}'_{zz} + \phi_2{}'_{\bar{z}\bar{z}}\right)\right]$$

$$- 2\beta^2\left[\left(\phi_{1zz} - \phi_{1\bar{z}\bar{z}}\right)\left(\phi_2{}'_z - \phi_2{}'_{\bar{z}}\right) + \left(\phi_{1z} - \phi_{1\bar{z}}\right)\left(\phi_2{}'_{zz} - \phi_2{}'_{\bar{z}\bar{z}}\right)\right]$$

$$+ 2\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]\left(\phi_{1z} + \phi_{1\bar{z}}\right)\phi_2{}''_{z\bar{z}}$$

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Now

$$\phi_2{}'_z = -6Ae^{-2\zeta} - 30Be^{-4\zeta}$$

$$\phi_2{}'_{zz} = \frac{12Ae^{-2\zeta} + 120Be^{-4\zeta}}{\sinh \zeta}$$

and

$$\phi_2{}'_z + \phi_2{}'_{\bar{z}} = -12Ae^{-2\xi}\cos 2\eta - 60Be^{-4\xi}\cos 4\eta$$

$$\phi_2{}'_z - \phi_2{}'_{\bar{z}} = 12iAe^{-2\xi}\sin 2\eta + 60iBe^{-4\xi}\sin 4\eta$$

$$\phi_2{}'_{zz} + \phi_2{}'_{\bar{z}\bar{z}} = \frac{1}{\sinh \zeta \sinh \bar{\zeta}}\Big[12A\left(e^{-\xi}\cos 3\eta - e^{-3\xi}\cos \eta\right)$$

$$+ 120B\left(e^{-3\xi}\cos 5\eta - e^{-5\xi}\cos 3\eta\right)\Big]$$

$$\phi_2{}'_{zz} - \phi_2{}'_{\bar{z}\bar{z}} = -\frac{i}{\sinh \zeta \sinh \bar{\zeta}}\Big[12A\left(e^{-\xi}\sin 3\eta - e^{-3\xi}\sin \eta\right)$$

$$+ 120B\left(e^{-3\xi}\sin 5\eta - e^{-5\xi}\sin 3\eta\right)\Big]$$

It follows that

$$T_2 \sinh \zeta \sinh \bar{\zeta} = \Bigg(-\frac{18}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right] Ae^{-3\xi}$$

$$+ \left\{\frac{9}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right](2A - 5B) + \frac{18}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]C\right\} e^{-5\xi}$$

$$+ \left\{\frac{90}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]B - \frac{6}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right](3C + 2D)\right\} e^{-7\xi}\Bigg) \cos \eta$$

$$+ \Bigg(\Bigg\{\frac{18}{\beta}(\gamma+1)\left(1-\beta^2\right)A - \frac{90}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]B$$

$$+ \frac{6}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right](3C + 2D)\Bigg\} e^{-5\xi}$$

$$+ \left\{\frac{45}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]B - \frac{18}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]C\right\} e^{-7\xi}\Bigg) \cos 3\eta$$

$$+ \Bigg(-\frac{18}{\beta}(\gamma+1)\left(1-\beta^2\right)Ae^{-3\xi}$$

$$+ \left\{\frac{135}{\beta}(\gamma+1)\left(1-\beta^2\right)B - \frac{12}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]D\right\} e^{-7\xi}\Bigg) \cos 5\eta$$

$$+ \left\{-\frac{135}{\beta}(\gamma+1)\left(1-\beta^2\right)B + \frac{12}{\beta}\left[(\gamma+1)\left(1+\beta^2\right) - 2\beta^2\right]D\right\} e^{-5\xi} \cos 7\eta \quad \text{(D-4)}$$

The calculation of T_1 and T_2 has not been very involved. The calculations for the remaining terms on the right-hand side of equation (D-1), however, are quite laborious and therefore only the final results will be presented. The terms to be calculated are

$$T_3 = \left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\left(\phi_{1zz} + \phi_{1\overline{zz}}\right)\left(\phi_2''{}_z + \phi_2''{}_{\overline{z}}\right)$$

$$T_4 = -2\beta^2\left(\phi_{1zz} - \phi_{1\overline{zz}}\right)\left(\phi_2''{}_z - \phi_2''{}_{\overline{z}}\right)$$

$$T_5 = \left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\left(\phi_{1z} + \phi_{1\overline{z}}\right)\left(\phi_2''{}_{zz} + \phi_2''{}_{\overline{zz}}\right)$$

$$T_6 = -2\beta^2\left(\phi_{1z} - \phi_{1\overline{z}}\right)\left(\phi_2''{}_{zz} - \phi_2''{}_{\overline{zz}}\right)$$

The expressions for these terms are as follows:

$$T_3 \sinh \zeta \sinh \overline{\zeta} = \frac{3}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\Big\{2C\Big[\left(2e^{\xi} - 3e^{-\xi} - 4e^{-3\xi} + 7e^{-5\xi} - 3e^{-7\xi}\right)\cos \eta$$

$$+\left(2e^{-\xi} - 3e^{-3\xi} + 3e^{-5\xi} - e^{-7\xi}\right)\cos 3\eta\Big] + D\Big[\left(4e^{3\xi} - 8e^{\xi} + 16e^{-3\xi} - 3e^{-5\xi} - 4e^{-7\xi}\right)\cos \eta$$

$$+\left(4e^{\xi} - 8e^{-\xi} - 4e^{-5\xi} - 5e^{-7\xi}\right)\cos 3\eta + \left(4e^{-\xi} - e^{-7\xi}\right)\cos 5\eta + \left(4e^{-3\xi} + e^{-5\xi}\right)\cos 7\eta\Big]\Big\}$$

$$-\frac{48}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\sinh^4 \xi \cosh \xi\left[C\left(2e^{-2\xi} + e^{-4\xi}\right) + 2D\right]\frac{\cos \eta}{\sinh \zeta \sinh \overline{\zeta}} \tag{D-5}$$

$$T_4 \sinh\zeta \sinh\bar{\zeta} = -6\beta\Big\{2C\Big[\left(2e^{\xi} - 3e^{-\xi} + e^{-5\xi}\right)\cos\eta + \left(2e^{-\xi} - 3e^{-3\xi} + e^{-7\xi}\right)\cos 3\eta\Big]$$

$$+ D\Big[\left(4e^{3\xi} - 8e^{\xi} - 3e^{-5\xi} + 4e^{-7\xi}\right)\cos\eta + \left(4e^{\xi} - 8e^{-\xi} + 4e^{-5\xi} - 5e^{-7\xi}\right)\cos 3\eta$$

$$+ \left(4e^{-\xi} + e^{-7\xi}\right)\cos 5\eta + \left(4e^{-3\xi} - e^{-5\xi}\right)\cos 7\eta\Big]\Big\}$$

$$+ 96\beta \sinh^4\xi \cosh\xi\left[C\left(2e^{-2\xi} + e^{-4\xi}\right) + 2D\right]\frac{\cos\eta}{\sinh\zeta\sinh\bar{\zeta}} \qquad \text{(D-6)}$$

$$T_5 \sinh\zeta \sinh\bar{\zeta} = \frac{3}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\Big\{C\Big[\left(-4e^{\xi} + 3e^{-\xi} + 11e^{-5\xi} - 15e^{-7\xi} + 5e^{-11\xi}\right)\cos\eta$$

$$+ \left(-3e^{-3\xi} + 3e^{-5\xi} - 3e^{-7\xi} + 3e^{-9\xi}\right)\cos 3\eta\Big] + 2D\Big[\left(-3e^{3\xi} + 4e^{\xi} + e^{-\xi} - 5e^{-5\xi}\right)\cos\eta$$

$$+ \left(-e^{\xi} + 3e^{-3\xi} - 3e^{-7\xi}\right)\cos 3\eta + e^{-\xi}\cos 5\eta + 3e^{-3\xi}\cos 7\eta\Big]\Big\}$$

$$+ \frac{6}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]\sinh^3\xi\Big[C\left(8 + 11e^{-2\xi} + 8e^{-4\xi} + 10e^{-6\xi} + 16e^{-8\xi} + 7e^{-10\xi}\right)$$

$$+ 2D\left(5e^{2\xi} + 4 + 2e^{-2\xi} + 4e^{-4\xi} + 5e^{-6\xi}\right)\Big]\frac{\cos\eta}{\sinh\zeta\sinh\bar{\zeta}}$$

$$- \frac{96}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 2\beta^2\right]e^{-2\xi}\sinh^5\xi\cosh^2\xi\cosh 2\xi\left[C\left(2e^{-2\xi} + e^{-4\xi}\right) + 2D\right]\frac{\cos\eta}{\sinh^2\zeta\sinh^2\bar{\zeta}} \qquad \text{(D-7)}$$

$$T_6 \sinh \zeta \sinh \bar{\zeta} = -6\beta \Bigg\{ C \bigg[\left(-4e^{\xi} + 3e^{-\xi} - 3e^{-5\xi} + 9e^{-7\xi} - 5e^{-11\xi} \right) \cos \eta$$

$$+ \left(-3e^{-3\xi} + 3e^{-5\xi} + 3e^{-7\xi} - 3e^{-9\xi} \right) \cos 3\eta \bigg] + 2D \bigg[\left(-3e^{3\xi} + 4e^{\xi} - e^{-\xi} + 3e^{-5\xi} \right) \cos \eta$$

$$+ \left(-e^{\xi} - 3e^{-3\xi} - 3e^{-7\xi} \right) \cos 3\eta + e^{-\xi} \cos 5\eta + 3e^{-3\xi} \cos 7\eta \bigg] \Bigg\}$$

$$- 48\beta \sinh^4\xi \cosh \xi \left[C\left(8e^{-2\xi} + 11e^{-4\xi} + 16e^{-6\xi} + 7e^{-8\xi} \right) + 2D\left(5 + 4e^{-2\xi} + 5e^{-4\xi} \right) \right] \frac{\cos \eta}{\sinh \zeta \sinh \bar{\zeta}}$$

$$+ 384\beta e^{-2\xi} \sinh^6\xi \cosh^3\xi \left[C\left(2e^{-2\xi} + e^{-4\xi} \right) + 2D \right] \frac{\cos \eta}{\sinh^2 \zeta \sinh^2 \bar{\zeta}} \tag{D-8}$$

Equation (D-1) can now be written as follows:

$$\frac{\partial^2 \phi_3}{\partial \xi^2} + \frac{\partial^2 \phi_3}{\partial \eta^2} = \frac{1 - \beta^2}{\beta^2} \bigg[\left(A_3{}^1 e^{-3\xi} + A_5{}^1 e^{-5\xi} + A_7{}^1 e^{-7\xi} + A_9{}^1 e^{-9\xi} + A_{11}{}^1 e^{-11\xi} + A_{13}{}^1 e^{-13\xi} \right) \cos \eta$$

$$+ \left(A_5{}^3 e^{-5\xi} + A_7{}^3 e^{-7\xi} + A_9{}^3 e^{-9\xi} + A_{11}{}^3 e^{-11\xi} + A_{15}{}^3 e^{-15\xi} \right) \cos 3\eta$$

$$+ \left(A_3{}^5 e^{-3\xi} + A_5{}^5 e^{-5\xi} + A_7{}^5 e^{-7\xi} + A_9{}^5 e^{-9\xi} + A_{11}{}^5 e^{-11\xi} + A_{13}{}^5 e^{-13\xi} + A_{17}{}^5 e^{-17\xi} \right) \cos 5\eta$$

$$+ \left(A_5{}^7 e^{-5\xi} + A_7{}^7 e^{-7\xi} + A_9{}^7 e^{-9\xi} + A_{11}{}^7 e^{-11\xi} + A_{13}{}^7 e^{-13\xi} + A_{15}{}^7 e^{-15\xi} + A_{19}{}^7 e^{-19\xi} \right) \cos 7\eta \bigg]$$

$$+ \frac{1-\beta^2}{\beta^2}\left\{\frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)\left[c\left(-12e^{-5\xi} + 21e^{-7\xi} - 9e^{-11\xi} + 20e^{-13\xi} - 33e^{-15\xi} + 13e^{-19\xi}\right)\right.\right.$$

$$\left. + 2D\left(-5e^{-3\xi} + 12e^{-5\xi} - 7e^{-7\xi} + 9e^{-11\xi} - 20e^{-13\xi} + 11e^{-15\xi}\right)\right]$$

$$+ 12\beta\left[c\left(-16e^{-9\xi} + 27e^{-11\xi} + 20e^{-13\xi} - 44e^{-15\xi} + 13e^{-19\xi}\right)\right.$$

$$\left.\left. + 2D\left(-7e^{-7\xi} + 16e^{-9\xi} - 20e^{-13\xi} + 11e^{-15\xi}\right)\right]\right\}\sum_{n=1}^{\infty} e^{-2n\xi}\cos(2n+7)\eta$$

$$+ \frac{1-\beta^2}{\beta^2}\left\{\frac{6}{\beta}(\gamma+1)\left(1-\beta^2\right)\left[c\left(-2e^{-5\xi} + 3e^{-7\xi} - e^{-11\xi} + 2e^{-13\xi} - 3e^{-15\xi} + e^{-19\xi}\right)\right.\right.$$

$$\left. + 2D\left(-e^{-3\xi} + 2e^{-5\xi} - e^{-7\xi} + e^{-11\xi} - 2e^{-13\xi} + e^{-15\xi}\right)\right]$$

$$+ 24\beta\left[c\left(-2e^{-9\xi} + 3e^{-11\xi} + 2e^{-13\xi} - 4e^{-15\xi} + e^{-19\xi}\right)\right.$$

$$\left.\left. + 2D\left(-e^{-7\xi} + 2e^{-9\xi} - 2e^{-13\xi} + e^{-15\xi}\right)\right]\right\}\sum_{n=1}^{\infty} ne^{-2n\xi}\cos(2n+7)\eta \qquad \text{(D-9)}$$

where expressions for the constants $A_q^{\,p}$ are given at the end of this appendix.

The right-hand side of equation (D-9) suggests a solution of the form

$$\phi_3 = G_1(\xi)\cos\eta + G_3(\xi)\cos 3\eta + G_5(\xi)\cos 5\eta + G_7(\xi)\cos 7\eta + \sum_{n=1}^{\infty} G_{2n+7}\cos(2n+7)\eta \qquad \text{(D-10)}$$

When this expression for ϕ_3 is inserted into equation (D-9) and the coefficients of $\cos\eta$, $\cos 3\eta$, . . . on both sides of the equation are equated, the following differential equations for G_1, G_3, G_5, G_7, and G_{2n+7} are obtained:

$$\frac{d^2G_1}{d\xi^2} - G_1 = \frac{1-\beta^2}{\beta^2}\left(A_3{}^1e^{-3\xi} + A_5{}^1e^{-5\xi} + A_7{}^1e^{-7\xi} + A_9{}^1e^{-9\xi} + A_{11}{}^1e^{-11\xi} + A_{13}{}^1e^{-13\xi}\right) \qquad \text{(D-11)}$$

$$\frac{d^2G_3}{d\xi^2} - 9G_3 = \frac{1-\beta^2}{\beta^2}\left(A_5{}^3e^{-5\xi} + A_7{}^3e^{-7\xi} + A_9{}^3e^{-9\xi} + A_{11}{}^3e^{-11\xi} + A_{15}{}^3e^{-15\xi}\right) \qquad \text{(D-12)}$$

$$\frac{d^2G_5}{d\xi^2} - 25G_5 = \frac{1-\beta^2}{\beta^2}\left(A_3{}^5e^{-3\xi} + A_5{}^5e^{-5\xi} + A_7{}^5e^{-7\xi} + A_9{}^5e^{-9\xi} + A_{11}{}^5e^{-11\xi} + A_{13}{}^5e^{-13\xi} + A_{17}{}^5e^{-17\xi}\right) \qquad \text{(D-13)}$$

$$\frac{d^2G_7}{d\xi^2} - 49G_7 = \frac{1-\beta^2}{\beta^2}\left(A_5{}^7e^{-5\xi} + A_7{}^7e^{-7\xi} + A_9{}^7e^{-9\xi} + A_{11}{}^7e^{-11\xi} + A_{13}{}^7e^{-13\xi} + A_{15}{}^7e^{-15\xi} + A_{19}{}^7e^{-19\xi}\right) \qquad \text{(D-14)}$$

$$
\begin{aligned}
\frac{d^2G_{2n+7}}{d\xi^2} - (2n+7)^2G_{2n+7} = \frac{1-\beta^2}{\beta^2}\Bigg\{\frac{3}{\beta}(\gamma+1)(1-\beta^2)\Big[C\left(-12e^{-5\xi}+21e^{-7\xi}-9e^{-11\xi}+20e^{-13\xi}-33e^{-15\xi}+13e^{-19\xi}\right)& \\
+ 2D\left(-5e^{-3\xi}+12e^{-5\xi}-7e^{-7\xi}+9e^{-11\xi}-20e^{-13\xi}+11e^{-15\xi}\right)\Big]& \\
+ 12\beta\Big[C\left(-16e^{-9\xi}+27e^{-11\xi}+20e^{-13\xi}-44e^{-15\xi}+13e^{-19\xi}\right)& \\
+ 2D\left(-7e^{-7\xi}+16e^{-9\xi}-20e^{-13\xi}+11e^{-15\xi}\right)\Big]\Bigg\}e^{-2n\xi}& \\
+ \frac{1-\beta^2}{\beta^2}\Bigg\{\frac{6}{\beta}(\gamma+1)(1-\beta^2)\Big[C\left(-2e^{-5\xi}+3e^{-7\xi}-e^{-11\xi}+2e^{-13\xi}-3e^{-15\xi}+e^{-19\xi}\right)& \\
+ 2D\left(-e^{-3\xi}+2e^{-5\xi}-e^{-7\xi}+e^{-11\xi}-2e^{-13\xi}+e^{-15\xi}\right)\Big]& \\
+ 24\beta\Big[C\left(-2e^{-9\xi}+3e^{-11\xi}+2e^{-13\xi}-4e^{-15\xi}+e^{-19\xi}\right)& \\
+ 2D\left(-e^{-7\xi}+2e^{-9\xi}-2e^{-13\xi}+e^{-15\xi}\right)\Big]\Bigg\}ne^{-2n\xi}& \qquad \text{(D-15)}
\end{aligned}
$$

The solutions of these equations are easily found and are as follows:

$$G_1(\xi) = \frac{1-\beta^2}{\beta^2}\left(C_1 e^{-\xi} + \frac{1}{8} A_3{}^1 e^{-3\xi} + \frac{1}{24} A_5{}^1 e^{-5\xi} + \frac{1}{48} A_7{}^1 e^{-7\xi}\right.$$

$$\left. + \frac{1}{80} A_9{}^1 e^{-9\xi} + \frac{1}{120} A_{11}{}^1 e^{-11\xi} + \frac{1}{168} A_{13}{}^1 e^{-13\xi}\right) \qquad \text{(D-16)}$$

$$G_3(\xi) = \frac{1-\beta^2}{\beta^2}\left(C_3 e^{-3\xi} + \frac{1}{16} A_5{}^3 e^{-5\xi} + \frac{1}{40} A_7{}^3 e^{-7\xi} + \frac{1}{72} A_9{}^3 e^{-9\xi}\right.$$

$$\left. + \frac{1}{112} A_{11}{}^3 e^{-11\xi} + \frac{1}{216} A_{15}{}^3 e^{-15\xi}\right) \qquad \text{(D-17)}$$

$$G_5(\xi) = \frac{1-\beta^2}{\beta^2}\left(C_5 e^{-5\xi} - \frac{1}{16} A_3{}^5 e^{-3\xi} - \frac{1}{10} A_5{}^5 \xi e^{-5\xi} + \frac{1}{24} A_7{}^5 e^{-7\xi} + \frac{1}{56} A_9{}^5 e^{-9\xi}\right.$$

$$\left. + \frac{1}{96} A_{11}{}^5 e^{-11\xi} + \frac{1}{144} A_{13}{}^5 e^{-13\xi} + \frac{1}{264} A_{17}{}^5 e^{-17\xi}\right) \qquad \text{(D-18)}$$

$$G_7(\xi) = \frac{1-\beta^2}{\beta^2}\left(C_7 e^{-7\xi} - \frac{1}{24} A_5{}^7 e^{-5\xi} - \frac{1}{14} A_7{}^7 \xi e^{-7\xi} + \frac{1}{32} A_9{}^7 e^{-9\xi} + \frac{1}{72} A_{11}{}^7 e^{-11\xi}\right.$$

$$\left. + \frac{1}{120} A_{13}{}^7 e^{-13\xi} + \frac{1}{176} A_{15}{}^7 e^{-15\xi} + \frac{1}{312} A_{19}{}^7 e^{-19\xi}\right) \qquad \text{(D-19)}$$

$$G_{2n+7}(\xi) = \frac{1-\beta^2}{\beta^2}\Bigg\{C_{2n+7} + \frac{3}{\beta}(\gamma+1)(1-\beta^2)\Bigg[C\left(\frac{1}{2}e^{2\xi} - \frac{3}{2}\xi - \frac{1}{8}e^{-4\xi} + \frac{1}{6}e^{-6\xi} - \frac{3}{16}e^{-8\xi} + \frac{1}{24}e^{-12\xi}\right)$$

$$+ 2D\left(\frac{1}{8}e^{4\xi} - \frac{1}{2}e^{2\xi} + \frac{1}{2}\xi + \frac{1}{8}e^{-4\xi} - \frac{1}{6}e^{-6\xi} + \frac{1}{16}e^{-8\xi}\right)\Bigg]$$

$$+ 12\beta\Bigg[C\left(-\frac{1}{2}e^{-2\xi} + \frac{3}{8}e^{-4\xi} + \frac{1}{6}e^{-6\xi} - \frac{1}{4}e^{-8\xi} + \frac{1}{24}e^{-12\xi}\right)$$

$$+ 2D\left(\frac{1}{2}\xi + \frac{1}{2}e^{-2\xi} - \frac{1}{6}e^{-6\xi} + \frac{1}{16}e^{-8\xi}\right)\Bigg]\Bigg\} e^{-(2n+7)\xi} \qquad \text{(D-20)}$$

The expression for the velocity potential ϕ can now be written as follows:

$$\phi = \cosh\xi\cos\eta + \frac{t}{4\beta}\left(3e^{-\xi}\cos\eta - e^{-3\xi}\cos 3\eta\right) + t^2\left(F_1\cos\eta + F_3\cos 3\eta + F_5\cos 5\eta\right)$$

$$+ t^3\Bigg[G_1\cos\eta + G_3\cos 3\eta + G_5\cos 5\eta + G_7\cos 7\eta$$

$$+ G_0\left(\frac{\sinh\xi\cos\eta}{\cosh 2\xi - \cos 2\eta} - e^{-\xi}\cos\eta - e^{-3\xi}\cos 3\eta - e^{-5\xi}\cos 5\eta - e^{-7\xi}\cos 7\eta\right)\Bigg] \qquad \text{(D-21)}$$

where F_1, F_3, and F_5 are given by equations (C-18) and G_0 is the following function of ξ:

$$G_0(\xi) = \frac{1-\beta^2}{\beta^2}\left\{C_0 + \frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)\left[C\left(\frac{1}{2}e^{2\xi} - \frac{3}{2}\xi - \frac{1}{8}e^{-4\xi} + \frac{1}{6}e^{-6\xi} - \frac{3}{16}e^{-8\xi} + \frac{1}{24}e^{-12\xi}\right)\right.\right.$$

$$\left. + 2D\left(\frac{1}{8}e^{4\xi} - \frac{1}{2}e^{2\xi} + \frac{1}{2}\xi + \frac{1}{8}e^{-4\xi} - \frac{1}{6}e^{-6\xi} + \frac{1}{16}e^{-8\xi}\right)\right]$$

$$+ 12\beta\left[C\left(-\frac{1}{2}e^{-2\xi} + \frac{3}{8}e^{4\xi} + \frac{1}{6}e^{-6\xi} - \frac{1}{4}e^{-8\xi} + \frac{1}{24}e^{-12\xi}\right)\right.$$

$$\left.\left. + 2D\left(\frac{1}{2}\xi + \frac{1}{2}e^{-2\xi} - \frac{1}{6}e^{-6\xi} + \frac{1}{16}e^{-8\xi}\right)\right]\right\} \qquad \text{(D-22)}$$

In this expression for G_0 the result has been anticipated that the arbitrary constants C_{2n+7} are independent of n and may be taken to be C_0. The constants C_0, C_1, C_3, C_5, and C_7 are determined by the boundary condition (equation (C-12)):

$$\left(\sinh\xi\cos\eta\frac{\partial\phi}{\partial\xi} - \cosh\xi\sin\eta\frac{\partial\phi}{\partial\eta}\right)\frac{dy}{dx} = \beta^2\left(\cosh\xi\sin\eta\frac{\partial\phi}{\partial\xi} + \sinh\xi\cos\eta\frac{\partial\phi}{\partial\eta}\right) \qquad \text{(C-12)}$$

If the expression for ϕ given by equation (D-21) is substituted into the boundary condition and if equations (C-14) and (C-15) are used, it follows that

$$
\left.
\begin{aligned}
C_0 &= -\frac{1}{16\beta}(\gamma+1)\left(1-\beta^2\right)(19C-34D)+\frac{1}{2}\beta(4C-19D)\\
C_1 &= \frac{9}{32}\frac{1}{1-\beta^2}\left[\frac{5}{2}\beta^3-\beta^2-\beta-\frac{1}{2\beta}-\frac{1}{2}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta^3-\frac{1}{8}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta\right]\\
&\quad -\left(\frac{3}{8}A_3{}^1+\frac{5}{24}A_5{}^1+\frac{7}{48}A_7{}^1+\frac{9}{80}A_9{}^1+\frac{11}{120}A_{11}{}^1+\frac{13}{168}A_{13}{}^1\right)\\
C_3 &= \frac{3}{16}\frac{1}{1-\beta^2}\left[-\frac{11}{4}\beta^3-\frac{5}{2}\beta^2+\frac{1}{4\beta}+1+\frac{3}{4}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta^3+\frac{1}{16}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta\right]\\
&\quad -\frac{1}{3}\left(\frac{5}{16}A_5{}^3+\frac{7}{40}A_7{}^3+\frac{1}{8}A_9{}^3+\frac{11}{112}A_{11}{}^3+\frac{5}{72}A_{15}{}^3\right)\\
C_5 &= \frac{3}{160}\frac{1}{1-\beta^2}\left[\frac{25}{2}\beta^3+31\beta^2+23\beta+\frac{9}{2\beta}+9-\frac{9}{2}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta^3+\frac{9}{8}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta\right]\\
&\quad -\frac{1}{5}\left(-\frac{3}{16}A_3{}^5+\frac{1}{10}A_5{}^5+\frac{7}{24}A_7{}^5+\frac{9}{56}A_9{}^5+\frac{11}{96}A_{11}{}^5+\frac{13}{144}A_{13}{}^5+\frac{17}{264}A_{17}{}^5\right)\\
C_7 &= \frac{3}{56}\frac{1}{1-\beta^2}\left[-\frac{7}{8}\beta^3-\frac{13}{4}\beta^2-5\beta-\frac{9}{8\beta}-\frac{15}{4}+\frac{3}{8}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta^3-\frac{9}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\beta\right]\\
&\quad -\frac{1}{7}\left(-\frac{5}{24}A_5{}^7+\frac{1}{14}A_7{}^7+\frac{9}{32}A_9{}^7+\frac{11}{72}A_{11}{}^7+\frac{13}{120}A_{13}{}^7+\frac{15}{176}A_{15}{}^7+\frac{19}{312}A_{19}{}^7\right)
\end{aligned}
\right\}
\quad \text{(D-23)}
$$

The expressions for the constants $A_q{}^p$ are as follows:

$$A_3{}^1 = \frac{6}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right](-3A - 4C + 8D)$$

$$A_5{}^1 = \frac{27}{64\beta^3}\left[(\gamma+1)\left(1-\beta^2\right)^2 + 8\beta^2\right] + \frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)(6A - 15B + 28C - 7D)$$

$$+ 12\beta\left[(3\gamma + 22)C + 6A - 15B - 8D\right]$$

$$A_7{}^1 = -\frac{27}{32\beta^3}(\gamma+1)\left(1-\beta^4\right) + \frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)(30B - 19C - 24D)$$

$$- 12\beta\left[\gamma(3C + 2D) - 30B + 7C + 20D\right]$$

$$A_9{}^1 = -\frac{15}{\beta}(\gamma+1)\left(1-\beta^2\right)(3C - 2D) - 120\beta(2C - D)$$

$$A_{11}{}^1 = \frac{15}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]C$$

$$A_{13}{}^1 = \frac{21}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]C$$

$$A_5{}^3 = \frac{27}{32\beta^3}(\gamma+1)\left(1-\beta^4\right) + \frac{9}{\beta}(\gamma+1)\left(1-\beta^2\right)(2A - 10B + 5C)$$

$$+ 12\beta\left[\gamma(3C + 2D) - 30B - 5C + 12D\right]$$

$$A_7{}^3 = -\frac{27}{64\beta^3}\left[(\gamma+1)\left(1-\beta^2\right)^2 + 8\beta^2\right] + \frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)(15B - 16C - D)$$

$$+ 12\beta\left[(-3\gamma + 10)C + 15B\right]$$

$$A_9{}^3 = \frac{9}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right](5C - 8D)$$

$$A_{11}{}^3 = -\frac{21}{\beta}(\gamma+1)\left(1-\beta^2\right)(3C - 2D) - 168\beta(2C - D)$$

$$A_{15}{}^3 = \frac{27}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]C$$

$$A_3{}^5 = -\frac{6}{\beta}(\gamma+1)\left(1-\beta^2\right)(3A + 4C - 8D)$$

$$A_5{}^5 = \frac{45}{\beta}(\gamma+1)\left(1-\beta^2\right)C - \frac{30}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]D$$

$$A_7{}^5 = -\frac{27}{64\beta^3}(\gamma+1)\left(1-\beta^2\right)^2 + \frac{15}{\beta}(\gamma+1)\left(1-\beta^2\right)(9B - D)$$

$$- 12\beta\left[12C + (2\gamma - 23)D\right]$$

$$A_9{}^5 = -\frac{21}{\beta}(\gamma+1)\left(1-\beta^2\right)(C - 2D) + 252\beta C$$

$$A_{11}{}^5 = \frac{48}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right](C - 2D)$$

$$A_{13}{}^5 = -\frac{27}{\beta}(\gamma+1)\left(1-\beta^2\right)(3C - 2D) - 216\beta(2C - D)$$

$$A_{17}{}^5 = \frac{33}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]C$$

$$A_5{}^7 = \frac{27}{64\beta^3}(\gamma+1)\left(1-\beta^2\right)^2 - \frac{3}{\beta}(\gamma+1)\left(1-\beta^2\right)(45B + 12C - 29D) + 12\beta(2\gamma + 1)D$$

$$A_7{}^7 = \frac{63}{\beta}(\gamma+1)\left(1-\beta^2\right)C - \frac{42}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]D$$

$$A_9{}^7 = -192\beta(C - 2D)$$

$$A_{11}{}^7 = -\frac{27}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) - 12\beta^2\right]C + \frac{54}{\beta}(\gamma+1)\left(1-\beta^2\right)D$$

$$A_{13}{}^7 = \frac{60}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right](C - 2D)$$

$$A_{15}{}^7 = -\frac{33}{\beta}\left[3(\gamma+1)\left(1-\beta^2\right) + 16\beta^2\right]C + \frac{66}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]D$$

$$A_{19}{}^7 = \frac{39}{\beta}\left[(\gamma+1)\left(1-\beta^2\right) + 4\beta^2\right]C$$

APPENDIX E

THE FLOW PAST A CURVED SURFACE BY THE POGGI METHOD

Equation (D-21) represents a solution of the fundamental differential equation (1) that satisfies the boundary conditions at the surface of the solid body and at infinity, insofar as the terms inclusive of the third power of the thickness coefficient t are concerned. The method used has been called Ackeret's iteration process and is valid for all values of the Mach number M from zero to unity. On the other hand, the method of Poggi yields the components of the fluid velocity in the form of power series in M^2. Since both Ackeret's and Poggi's methods provide solutions of equation (1), the two solutions must agree in the region common to both. The flow past the shape treated in the present paper will be calculated by means of Poggi's method and compared with that obtained by Ackeret's method.

Poggi's method consists in regarding a compressible fluid as an incompressible fluid with a continuous distribution of sources in the region external to the solid boundary. In order to express the intensity of the source distribution, it is first necessary to determine the incompressible flow, which serves as the zero approximation to the flow of the compressible fluid. The first-order effect of compressibility on the velocity of the fluid is then given by a set of double integrals extended over the entire region of flow. In reference 5 the surface integrals are replaced by line integrals, which are evaluated by the methods of the calculus of residues. For the example treated herein, the general results given in reference 5 are immediately applicable and are as follows:

Let $Z = Z(Z')$ be the conformal transformation of the profile in the Z plane into the circle of radius c with center at the origin of the Z' plane. Then, if W_o is the complex velocity of an incompressible fluid past the circle, the following expressions are formed:

$$\left.\begin{aligned} F(Z',\overline{Z}') &= W_o{}^2\overline{W}_o \frac{dZ'}{dZ}\frac{d\overline{Z}'}{d\overline{Z}} + \frac{d}{dZ'}\left(W_o \frac{dZ'}{dZ}\right)\int \overline{W}_o{}^2 \frac{d\overline{Z}'}{d\overline{Z}}\, d\overline{Z}' \\ \overline{F}(\overline{Z}',Z') &= \overline{W}_o{}^2 W_o \frac{dZ'}{dZ}\frac{d\overline{Z}'}{d\overline{Z}} + \frac{d}{d\overline{Z}'}\left(\overline{W}_o \frac{d\overline{Z}'}{d\overline{Z}}\right)\int W_o{}^2 \frac{dZ'}{dZ}\, dZ' \end{aligned}\right\} \qquad \text{(E-1)}$$

The complex velocity W of the compressible fluid past the profile in the Z plane is given by

$$W = \left(W_o + W_1 M^2\right) \frac{dZ'}{dZ} \tag{E-2}$$

where only the terms involving the square of the Mach number M are shown. The expression for W_1 is as follows:

$$W_1 = -\frac{1}{4U^2}\left[S(Z') - \frac{c^2}{Z'^2}\,\overline{S}\left(\frac{c^2}{Z'}\right) - \frac{c^2}{Z'^2}\,\overline{F}\left(\frac{c^2}{Z'}, Z'\right)\right]$$

$$-\frac{1}{4U^2} F(Z', \overline{Z}') + \frac{1}{4} U\left(1 - \frac{c^2}{Z'^2}\right) \tag{E-3}$$

where $\overline{F}\left(\frac{c^2}{Z'}, Z'\right)$ is obtained from $\overline{F}(\overline{Z}', Z')$ by replacing $\overline{Z}'$ by $\frac{c^2}{Z'}$; $S(Z')$ denotes the sum of the residues of $\frac{1}{Z' - Z'_P} F\left(Z', \frac{c^2}{Z'}\right)$ at the poles within the circle C_1 of radius c, the subscript P being dropped after the evaluation; and $\overline{S}\left(\frac{c^2}{Z'}\right)$ is obtained from the expression for $\overline{S}(\overline{Z}')$ by replacing $\overline{Z}'$ by $\frac{c^2}{Z'}$.

For the present example, according to equations (A-3), (A-4), and (A-6),

$$\frac{dZ}{dZ'} = \left(1 + \frac{c}{Z'}\right)\left(1 - \frac{c}{Z'}\right)\left(1 + \frac{id}{Z'}\right)\left(1 - \frac{id}{Z'}\right) \tag{A-3}$$

$$Z = Z' + \frac{c^2 - d^2}{Z'} + \frac{c^2 d^2}{3Z'^3} \tag{A-4}$$

and

$$W_o = U\left(1 - \frac{c^2}{Z'^2}\right)$$

It follows that

$$F(Z',\overline{Z}') = U^3 \frac{1 - \frac{c^2}{Z'^2}}{\left(1 - \frac{id}{Z'}\right)\left(1 + \frac{id}{Z'}\right)\left(1 - \frac{id}{\overline{Z}'}\right)\left(1 + \frac{id}{\overline{Z}'}\right)}$$

$$+ 2U^3d^2 \frac{1}{Z'^3\left(1 - \frac{id}{Z'}\right)^2\left(1 + \frac{id}{Z'}\right)^2} \left[\overline{Z}' + \frac{1}{2id}\left(c^2 + d^2\right) \log \frac{\overline{Z}' + id}{\overline{Z}' - id}\right] \tag{E-4}$$

and

$$F\left(Z', \frac{c^2}{Z'}\right) = \frac{U^3c^4}{d^2} \frac{Z'^2 - c^2}{(Z' - id)(Z' + id)\left(Z' - \frac{ic^2}{d}\right)\left(Z' + \frac{ic^2}{d}\right)}$$

$$+ 2U^3d^2 \frac{Z'}{(Z' - id)^2(Z' + id)^2} \left[\frac{c^2}{Z'} + \frac{1}{2id}\left(c^2 + d^2\right) \log \frac{\frac{ic^2}{d} - Z'}{\frac{ic^2}{d} + Z'}\right] \tag{E-5}$$

Consider now the contour integral

$$\int_{C_1} \frac{1}{Z' - Z'_P} F\left(Z', \frac{c^2}{Z'}\right) dZ' = 2\pi i S(Z'_P)$$

In the first term on the right-hand side of equation (E-5) only the simple poles at $Z' = \pm id$ are internal to the circle C_1. The contributions of these simple poles to the residue are, at $Z' = id$,

$$\frac{U^3c^4}{2id\left(c^2 - d^2\right)} \frac{1}{Z'_P - id}$$

and, at $Z' = -id$,

$$-\frac{U^3c^4}{2id\left(c^2 - d^2\right)} \frac{1}{Z'_P + id}$$

If the two expressions are added and the subscript P is dropped, the result is

$$\frac{U^3c^4}{c^2 - d^2} \frac{1}{Z'^2 + d^2} \tag{E-6}$$

The second term on the right-hand side of equation (E-5) can be rewritten as follows:

$$2U^3 \frac{d^2c^2}{(Z' - id)^2(Z' + id)^2} - iU^3d\left(c^2 + d^2\right) \frac{Z'}{(Z' - id)^2(Z' + id)^2} \log \frac{\frac{ic^2}{d} - Z'}{\frac{ic^2}{d} + Z'}$$

The contributions of the double poles at $Z' = id$ and $Z' = -id$ to the residue are, respectively,

$$-\frac{U^3c^2}{2id} \frac{Z'_P - 2id}{(Z'_P - id)^2} + \frac{1}{2} \frac{iU^3c^2d}{c^2 - d^2} \frac{1}{Z'_P - id}$$

$$+ \frac{1}{4} U^3\left(c^2 + d^2\right) \frac{1}{(Z'_P - id)^2} \log \frac{c^2 - d^2}{c^2 + d^2}$$

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and

$$\frac{U^3c^2}{2id}\frac{Z'_P + 2id}{(Z'_P + id)^2} - \frac{1}{2}\frac{iU^3c^2d}{c^2 - d^2}\frac{i}{Z'_P + id}$$

$$+ \frac{1}{4}U^3\left(c^2 + d^2\right)\frac{1}{(Z'_P + id)^2}\log\frac{c^2 - d^2}{c^2 + d^2}$$

If the two expressions are added and the subscript P is dropped, the result is

$$-\frac{2U^3c^2d^2}{\left(Z'^2 + d^2\right)^2} - \frac{U^3c^2d^2}{c^2 - d^2}\frac{1}{Z'^2 + d^2}$$

$$+ \frac{1}{2}U^3\left(c^2 + d^2\right)\frac{Z'^2 - d^2}{\left(Z'^2 + d^2\right)^2}\log\frac{c^2 - d^2}{c^2 + d^2} \qquad \text{(E-7)}$$

If the expressions (E-6) and (E-7) are added, it follows that

$$S(Z') = U^3c^2\left(1 + \frac{c^2 + d^2}{2c^2}\log\frac{c^2 - d^2}{c^2 + d^2}\right)\frac{Z'^2 - d^2}{\left(Z'^2 + d^2\right)^2} \qquad \text{(E-8)}$$

From equation (E-4),

$$F(Z',\bar{Z}') = U^3\frac{\left(Z'^2 - c^2\right)Z'^2\bar{Z}'^2}{Z'^2\left(Z'^2 + d^2\right)\left(\bar{Z}'^2 + d^2\right)}$$

$$+ \frac{2U^3d^2}{\left(Z'^2 + d^2\right)^2}\left(Z'\bar{Z}' + \frac{c^2 + d^2}{2id}Z'\log\frac{\bar{Z}' + id}{\bar{Z}' - id}\right) \qquad \text{(E-9)}$$

If $\bar{S}(\bar{Z}')$ is formed from equation (E-8) and $\bar{Z}'$ is replaced by $\frac{c^2}{Z'}$, it follows that

$$\overline{S}\left(\frac{c^2}{Z'}\right) = U^3c^2\left(1 + \frac{c^2 + d^2}{2c^2}\log\frac{c^2 - d^2}{c^2 + d^2}\right)\frac{\left(c^4 - d^2Z'^2\right)Z'^2}{\left(c^4 + d^2Z'^2\right)^2} \qquad \text{(E-10)}$$

Finally, if $\overline{F}(\overline{Z}',Z')$ is formed from equation (E-9) and $\overline{Z}'$ is replaced by c^2/Z', it follows that

$$\overline{F}\left(\frac{c^2}{Z'},Z'\right) = U^3c^2\frac{Z'^2\left(c^2 - Z'^2\right)}{\left(c^4 + d^2Z'^2\right)\left(d^2 + Z'^2\right)}$$

$$+ 2U^3d^2c^2\frac{Z'^3}{\left(c^4 + d^2Z'^2\right)^2}\left(Z' + \frac{c^2 + d^2}{2id}\log\frac{Z' + id}{Z' - id}\right) \qquad \text{(E-11)}$$

By means of equations (E-8), (E-9), (E-10), and (E-11) and by replacing Z'/c, $\overline{Z}'/c$, W_1/U, and d^2/c^2 by Z', $\overline{Z}'$, W_1, and ϵ, respectively, it follows that

$$W_1 = \frac{1}{4}\left\{\left(1 - \frac{1+\epsilon}{2}\log\frac{1+\epsilon}{1-\epsilon}\right)\left[\frac{Z'^2 - \epsilon}{\left(Z'^2 + \epsilon\right)^2} + \frac{\epsilon Z'^2 - 1}{\left(\epsilon Z'^2 + 1\right)^2}\right]\right.$$

$$+ \frac{Z'^2 - 1}{\left(\epsilon + Z'^2\right)\left(\epsilon Z'^2 + 1\right)} - \frac{2\epsilon}{\left(\epsilon Z'^2 + 1\right)^2}\left[Z'^2 + (1+\epsilon)\frac{Z'}{2i\sqrt{\epsilon}}\log\frac{Z' + i\sqrt{\epsilon}}{Z' - i\sqrt{\epsilon}}\right]$$

$$- 1 + \frac{1}{Z'^2} + \frac{\overline{Z}'^2\left(Z'^2 - 1\right)}{\left(Z'^2 + \epsilon\right)\left(\overline{Z}'^2 + \epsilon\right)}$$

$$\left. + \frac{2\epsilon}{\left(Z'^2 + \epsilon\right)^2}\left[Z'\overline{Z}' + (1+\epsilon)\frac{Z'}{2i\sqrt{\epsilon}}\log\frac{\overline{Z}' + i\sqrt{\epsilon}}{Z' - i\sqrt{\epsilon}}\right]\right\} \qquad \text{(E-12)}$$

The complex velocity W of the compressible fluid past the profile in the Z plane is given by equation (E-2) where

$$\frac{dZ'}{dZ} = \frac{Z'^4}{\left(Z'^2 - 1\right)\left(Z'^2 + \epsilon\right)} \qquad \text{(from equation (A-3))} \qquad \text{(E-13)}$$

$$W_o = 1 - \frac{1}{Z'^2} \qquad \text{(from equation (A-6))} \qquad \text{(E-14)}$$

and W_1 is given by equation (E-12).

The velocity at the surface of the profile is obtained by means of equations (E-2), (E-12), (E-13), and (E-14) with $Z' = e^{i\theta}$ and $\overline{Z}' = e^{-i\theta}$. Thus

$$q_{profile} = \frac{1}{\sqrt{1+2\epsilon\cos 2\theta+\epsilon^2}}\left(1+\frac{1}{1+2\epsilon\cos 2\theta+\epsilon^2}\frac{M^2}{4}\left\{\epsilon(1-\epsilon-2\cos 2\theta)\right.\right.$$

$$-\frac{(1+\epsilon)^2}{2}\left[1-\frac{2(1-\epsilon)^2}{1+2\epsilon\cos 2\theta+\epsilon^2}\right]\log\frac{1+\epsilon}{1-\epsilon}$$

$$+\frac{(1+\epsilon)}{2\sqrt{\epsilon}}\,\frac{1}{1+2\epsilon\cos 2\theta+\epsilon^2}\left[\frac{2\epsilon+(1+\epsilon^2)\cos 2\theta}{\sin\theta}\log\frac{1+\epsilon+2\sqrt{\epsilon}\sin\theta}{1+\epsilon-2\sqrt{\epsilon}\sin\theta}\right.$$

$$\left.\left.\left.-4(1-\epsilon^2)\cos\theta\tan^{-1}\frac{2\sqrt{\epsilon}\cos\theta}{1-\epsilon}\right]\right\}\right) \qquad \text{(E-15)}$$

For $\theta = 0$,

$$q_{profile} = \frac{1}{1+\epsilon}\left(1+\frac{M^2}{4}\left\{\frac{\epsilon(1-\epsilon)}{(1+\epsilon)^2}+\left[\left(\frac{1-\epsilon}{1+\epsilon}\right)^2-\frac{1}{2}\right]\log\frac{1+\epsilon}{1-\epsilon}\right.\right.$$

$$\left.\left.-\frac{2\sqrt{\epsilon}(1-\epsilon)}{(1+\epsilon)^2}\tan^{-1}\frac{2\sqrt{\epsilon}}{1-\epsilon}\right\}\right) \qquad \text{(E-16)}$$

For $\theta = \frac{\pi}{2}$,

$$q_{profile} = \frac{1}{1 - \epsilon}\left\{1 + \frac{M^2}{4}\left[\frac{1 + \epsilon}{(1 - \epsilon)^2} - 1 + \frac{1}{2}\left(\frac{1 + \epsilon}{1 - \epsilon}\right)^2 \log \frac{1 + \epsilon}{1 - \epsilon}\right.\right.$$

$$\left.\left. - \frac{\sqrt{\epsilon}(1 + \epsilon)}{(1 - \epsilon)^2} \log \frac{1 + \sqrt{\epsilon}}{1 - \sqrt{\epsilon}}\right]\right\} \tag{E-17}$$

The velocity along the X axis external to the profile is obtained from equations (E-2), (E-12), (E-13), and (E-14) by taking $Z' = \bar{Z}' = X'$. Thus

$$q_{Y=0} = \frac{X'^2}{X'^2 + \epsilon}\left\{1 + \frac{M^2}{4}\left[(1 + \epsilon)\left(1 - \frac{1 + \epsilon}{2} \log \frac{1 + \epsilon}{1 - \epsilon}\right)\frac{\epsilon X'^6 - (\epsilon^2 - 4\epsilon + 1)X'^4 + \epsilon X'^2}{(X'^2 + \epsilon)^2(\epsilon X'^2 + 1)^2}\right.\right.$$

$$+ \frac{X'^2}{(X'^2 + \epsilon)(\epsilon X'^2 + 1)} - 1 + \frac{X'^4}{(X'^2 + \epsilon)^2}$$

$$\left.\left. - \frac{2\epsilon(1 - \epsilon^2)X'^3(X'^2 + 1)}{(X'^2 + \epsilon)^2(\epsilon X'^2 + 1)^2}\left(X' + \frac{1 + \epsilon}{2\sqrt{\epsilon}} \tan^{-1} \frac{2\sqrt{\epsilon}\, X'}{X'^2 - \epsilon}\right)\right]\right\} \tag{E-18}$$

The velocity along the Y axis external to the profile is obtained from equations (E-2), (E-12), (E-13), and (E-14) by taking $Z' = iY'$ and $\bar{Z}' = -iY'$. Thus

$$q_{X=0} = \frac{Y'^2}{Y'^2 - \epsilon}\left\{1 + \frac{M^2}{4}\left[-(1+\epsilon)\left(1 - \frac{1+\epsilon}{2}\log\frac{1+\epsilon}{1-\epsilon}\right)\frac{\epsilon Y'^6 + \left(\epsilon^2 - 4\epsilon + 1\right)Y'^4 + \epsilon Y'^2}{\left(\epsilon - Y'^2\right)^2\left(1 - \epsilon Y'^2\right)^2}\right.\right.$$

$$- \frac{Y'^2}{\left(\epsilon - Y'^2\right)\left(1 - \epsilon Y'^2\right)} - 1 + \frac{Y'^4}{\left(\epsilon - Y'^2\right)^2}$$

$$\left.\left. + 2\epsilon Y'^3\left(Y' - \frac{1+\epsilon}{2\sqrt{\epsilon}}\log\frac{Y' + \sqrt{\epsilon}}{Y' - \sqrt{\epsilon}}\right)\frac{\left(1+\epsilon^2\right)\left(Y'^2 + 1\right)^2 - 2(1+\epsilon)^2 Y'^2}{\left(Y'^2 + 1\right)\left(1 - \epsilon Y'^2\right)\left(\epsilon - Y'^2\right)^2}\right]\right\} \qquad \text{(E-19)}$$

Table III gives values of the velocity corresponding to points along the profile for the numerical case $\epsilon = \frac{1}{7}$ (or $t = 0.10$) at Mach numbers $M = 0.50$ and 0.75. Figure 1 shows a comparison of the velocity distribution for $M = 0.50$ with the corresponding calculation according to the iteration method.

It is noted that by Poggi's method the solution of the problem is given by the components of the fluid velocity, whereas by Ackeret's method the solution is obtained in the form of the velocity potential of the fluid motion. Before a comparison of the two methods can be made, therefore, it is necessary to obtain from the velocity potential (of Ackeret) the velocity components along the coordinate axes. This calculation is performed in appendix F.

APPENDIX F

DETERMINATION OF THE VELOCITY COMPONENTS

In terms of the complex variables ζ and $\bar{\zeta}$ the velocity components in the direction of the coordinate axes are given by

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial X} = \frac{1}{\sinh \zeta}\frac{\partial \phi}{\partial \zeta} + \frac{1}{\sinh \bar{\zeta}}\frac{\partial \phi}{\partial \bar{\zeta}} \\ v &= \frac{\partial \phi}{\partial Y} = i\beta\left(\frac{1}{\sinh \zeta}\frac{\partial \phi}{\partial \zeta} - \frac{1}{\sinh \bar{\zeta}}\frac{\partial \phi}{\partial \bar{\zeta}}\right) \end{aligned}\right\} \quad \text{(F-1)}$$

where the velocity potential ϕ is obtained from equation (D-21). According to equations (6)

$$\left.\begin{aligned} u &= 1 + tu_1 + t^2u_2 + t^3u_3 + \dots \\ v &= tv_1 + t^2v_2 + t^3v_3 + \dots \end{aligned}\right\} \quad \text{(F-2)}$$

where, in general,

$$u_n = \frac{\partial \phi_n}{\partial X}$$

$$v_n = \frac{\partial \phi_n}{\partial Y}$$

In terms of the complex variables ζ and $\bar{\zeta}$,

$$\phi_1 = \frac{1}{8\beta}\left(3e^{-\zeta} - e^{-3\zeta} + 3e^{-\bar{\zeta}} - e^{-3\bar{\zeta}}\right)$$

Therefore,

$$\left.\begin{aligned} u_1 &= -\frac{3}{2\beta}e^{-2\xi}\cos 2\eta \\ v_1 &= -\frac{3}{2}e^{-2\xi}\sin 2\eta \end{aligned}\right\} \quad \text{(F-3)}$$

Similarly, with reference to equations (D-3), it follows that:

$$\left.\begin{aligned}
u_2 &= 4C\left(2 - 3e^{-2\xi} + 3e^{-4\xi} - e^{-6\xi}\right) + 8D\left(e^{2\xi} - 2 + e^{-2\xi}\right) \\
&\quad + 4\left(4D - 3Ae^{-2\xi} + 4De^{-4\xi}\right)\cos 2\eta + 4\left[4De^{-2\xi} + (D - 15B)e^{-4\xi}\right]\cos 4\eta \\
&\quad - \frac{32 \sinh^3\xi \cosh\xi \left[2D + C\left(2e^{-2\xi} + e^{-4\xi}\right)\right]}{\cosh 2\xi - \cos 2\eta} \\
v_2 &= 4\beta \sin 2\eta \left\{\left(4D - 3Ae^{-2\xi} + 4De^{-4\xi}\right)\right. \\
&\quad + 2\left[4De^{-2\xi} - (D + 15B)e^{-4\xi}\right]\cos 2\eta \\
&\quad \left. - \frac{4 \sinh^2\xi \left[2D + C\left(2e^{-2\xi} + e^{-4\xi}\right)\right]}{\cosh 2\xi - \cos 2\eta}\right\}
\end{aligned}\right\} \quad \text{(F-4)}$$

The general expressions for the velocity components u_3 and v_3 are too cumbersome to be given here. Instead, only the expressions for the velocity components along the profile will be given. Thus, along the profile, if powers of t higher than the third are neglected,

$$\left.\begin{aligned}
u_3 &= \left(G_1 + 3G_3 + 5G_5 + 7G_7\right)_0 + 2\left(3G_3 + 5G_5 + 7G_7\right)_0 \cos 2a \\
&\quad + 2\left(5G_5 + 7G_7\right)_0 + 14\left(G_7\right)_0 \cos 6a \\
v_3 &= -2\beta\left[\left(G_3' + G_5' + G_7'\right)_0 \sin 2a \right. \\
&\quad \left. + \left(G_5' + G_7'\right)_0 \sin 4a + \left(G_7'\right)_0 \sin 6a\right]
\end{aligned}\right\} \quad \text{(F-5)}$$

where the primes denote differentiation with regard to the independent variable ξ and the zero subscripts denote evaluation for $\xi = 0$. Explicit expressions for $(G_1)_0$, $(G_3)_0$, $(G_5)_0$, $(G_7)_0$, $(G_1')_0$, $(G_3')_0$, $(G_5')_0$, and $(G_7')_0$ are given at the end of this appendix.

At the boundary then, if terms containing powers of t higher than the third are neglected, equations (F-2) become

$$\left.\begin{aligned}
u = 1 &+ \left[\frac{3}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 + \frac{3}{8}\,\frac{1-3\beta^2}{\beta^2}\right]t^2 + \left[\frac{3}{16}(2+\beta) + \left(G_1 + 3G_3 + 5G_5 + 7G_7\right)_0\right]t^3 \\
&+ \left\{-\frac{3}{2\beta}t + \frac{3}{4\beta}t^2 + \left[\frac{9}{32}\,\frac{4+2\beta+3\beta^2}{\beta} + 2\left(3G_3 + 5G_5 + 7G_7\right)_0\right]t^3\right\}\cos 2\alpha \\
&+ \left\{\left[\frac{9}{32}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 + \frac{3}{8}\,\frac{3+5\beta}{\beta^2}\right]t^2 + \left[-\frac{9}{16}\,\frac{4+6\beta+3\beta^2}{\beta} + 2\left(5G_5 + 7G_7\right)_0\right]t^3\right\}\cos 4\alpha \\
&+ \left[\frac{3}{32}\,\frac{12+26\beta+7\beta^2}{\beta} + 14\left(G_7\right)_0\right]t^3\cos 6\alpha \\
v = &\left[-\frac{3}{2}t + \frac{3}{4}t^2 + \frac{9}{32}\left(4-2\beta+5\beta^2\right)t^3 - \frac{9}{32}(\gamma+1)\frac{\left(1-\beta^2\right)^2}{\beta^2}t^3 - 2\beta\left(G_3' + G_5' + G_7'\right)_0 t^3\right]\sin 2\alpha \\
&+ \left[\frac{3}{8}\,\frac{3+5\beta}{\beta}t^2 - \frac{9}{16}\left(4+6\beta+3\beta^2\right)t^3 + \frac{9}{16}(\gamma+1)\frac{\left(1-\beta^2\right)^2}{\beta^2}t^3 - 2\beta\left(G_5' + G_7'\right)_0 t^3\right]\sin 4\alpha \\
&+ \left[\frac{3}{32}\left(12+26\beta+7\beta^2\right) - \frac{9}{32}(\gamma+1)\frac{\left(1-\beta^2\right)^2}{\beta^2} - 2\beta\left(G_7'\right)_0\right]t^3\sin 6\alpha
\end{aligned}\right\} \quad \text{(F-6)}$$

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In order to compare the velocity of the fluid given by equations (F-6) with that obtained by Poggi's method given by equation (E-15), explicit expressions for equations (F-6), with powers of the Mach number higher than the second neglected, are obtained and may be expressed as follows:

$$\left.\begin{aligned}
u &= \left(1 - \frac{3}{4} t^2 + \frac{9}{16} t^3\right) + \left(-\frac{3}{2} t + \frac{3}{4} t^2 + \frac{81}{32} t^3\right) \cos 2\alpha \\
&\quad + \left(3t^2 - \frac{45}{16} t^3\right) \cos 4\alpha - \frac{201}{32} t^3 \cos 6\alpha \\
&\quad + M^2 \left[\left(\frac{3}{8} t^2 - \frac{111}{160} t^3\right) + \left(-\frac{3}{4} t + \frac{3}{8} t^2 - \frac{369}{320} t^3\right) \cos 2\alpha \right. \\
&\quad \left. + \left(\frac{33}{16} t^2 - \frac{63}{32} t^3\right) \cos 4\alpha - \frac{345}{64} t^3 \cos 6\alpha \right] \\
v &= \left(-\frac{3}{2} t + \frac{3}{4} t^2 + \frac{63}{32} t^3\right) \sin 2\alpha + \left(3t^2 - \frac{45}{16} t^3\right) \sin 4\alpha - \frac{201}{32} t^3 \sin 6\alpha \\
&\quad + M^2 \left[\frac{9}{32} t^3 \sin 2\alpha + \left(\frac{9}{16} t^2 - \frac{9}{4} t^3\right) \sin 4\alpha - \frac{9}{4} t^3 \sin 6\alpha \right]
\end{aligned}\right\} \quad \text{(F-7)}$$

The magnitude q of the velocity, to the same degree of approximation, is then given by

$$\begin{aligned}
q &= \left(1 - \frac{3}{16} t^2\right) + \left(-\frac{3}{2} t + \frac{3}{4} t^2 + \frac{45}{64} t^3\right) \cos 2\alpha + \left(\frac{39}{16} t^2 - \frac{9}{4} t^3\right) \cos 4\alpha \\
&\quad - \frac{285}{64} t^3 \cos 6\alpha + M^2 \left[\left(\frac{3}{8} t^2 - \frac{111}{160} t^3\right) + \left(-\frac{3}{4} t + \frac{3}{8} t^2 - \frac{873}{640} t^3\right) \cos 2\alpha \right. \\
&\quad \left. + \left(\frac{33}{16} t^2 - \frac{63}{32} t^3\right) \cos 4\alpha - \frac{663}{128} t^3 \cos 6\alpha \right]
\end{aligned} \qquad \text{(F-8)}$$

which is to be compared with the corresponding expression obtained by Poggi's method. Thus, if in equation (E-15) ϵ is replaced by $3t/(2+t)$ and all terms are expanded in powers of t up to and including t^3, the resulting equation is as follows:

$$q_{profile} = \left(1 + \frac{9}{16}t^2 - \frac{9}{16}t^3\right) - \left(\frac{3}{2}t - \frac{3}{4}t^2 + \frac{105}{64}t^3\right)\cos 2\theta$$

$$+ \frac{27}{16}\left(t^2 - t^3\right)\cos 4\theta - \frac{135}{64}t^3\cos 6\theta$$

$$+ M^2\left[\left(\frac{3}{4}t^2 - \frac{39}{40}t^3\right) + \left(-\frac{3}{4}t + \frac{3}{8}t^2 - \frac{2163}{640}t^3\right)\cos 2\theta\right.$$

$$\left. + \frac{27}{16}\left(t^2 - t^3\right)\cos 4\theta - \frac{405}{128}t^3\cos 6\theta\right] \qquad \text{(F-9)}$$

From the first of equations (A-9) with $X = \cos\alpha$, it follows that

$$\cos\alpha = (1 - t\sin^2\theta)\cos\theta$$

and therefore

$$\cos 2\alpha = \cos 2\theta - \sin^2 2\theta\left(t - \frac{1}{2}t^2\sin^2\theta\right)$$

$$\cos 4\alpha = \cos 4\theta - 2t\sin 4\theta\sin 2\theta$$

$$\cos 6\alpha = \cos 6\theta$$

If $\cos 2\alpha$, $\cos 4\alpha$, and $\cos 6\alpha$ in equation (F-8) are replaced by these expressions, equation (F-9) is again obtained.

The expressions for $(G_1)_0$, $(G_3)_0$, $(G_5)_0$, $(G_7)_0$, $(G_1')_0$, $(G_3')_0$, $(G_5')_0$, and $(G_7')_0$ are as follows:

$$(G_1)_0 = \frac{9}{32}\left[\frac{5}{2}\beta - 1 - \frac{1}{\beta} - \frac{1}{2\beta^3} - \frac{1}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta - \frac{1}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$- \frac{M^2}{\beta^2}\left[-\frac{27}{256\beta^3}(\gamma+1)\left(1-\beta^4\right) + \frac{9}{128}(\gamma+1)\frac{M^4}{\beta^3} + \frac{9}{16\beta} + 3\gamma\beta\left(\frac{1}{2}C - D\right)\right.$$

$$\left. + (\gamma+1)\frac{M^2}{\beta}\left(-\frac{3}{2}A + \frac{15}{4}B - \frac{7}{8}C + \frac{5}{2}D\right) + \beta\left(-6A + 15B - \frac{7}{2}C + 14D\right)\right]$$

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$$\left(G_3\right)_0 = \frac{3}{16}\left[-\frac{11}{4}\beta - \frac{5}{2} + \frac{1}{4\beta^3} + \frac{1}{\beta^2} + \frac{3}{4}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta + \frac{1}{16}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$- \frac{M^2}{\beta^2}\left[\frac{9}{256\beta^3}(\gamma+1)\left(1-\beta^4\right) - \frac{9}{640}(\gamma+1)\frac{M^4}{\beta^3} - \frac{9}{80\beta} + \gamma\beta\left(\frac{3}{10}C + D\right)\right.$$

$$\left. + (\gamma+1)\frac{M^2}{\beta}\left(\frac{3}{4}A - \frac{9}{4}B + \frac{21}{40}C - \frac{11}{10}D\right) + \beta\left(-9B + \frac{1}{2}C + 2D\right)\right]$$

$$\left(G_5\right)_0 = \frac{3}{160}\left[\frac{25}{2}\beta + 31 + \frac{23}{\beta} + \frac{9}{2\beta^3} + \frac{9}{\beta^2} - \frac{9}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta + \frac{9}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$- \frac{M^2}{\beta^2}\left[-\frac{9}{1280}(\gamma+1)\frac{M^4}{\beta^3} - \frac{2}{5}\gamma\beta D + (\gamma+1)\frac{M^2}{\beta}\left(-\frac{9}{20}A + \frac{9}{4}B + \frac{7}{20}D\right) - \frac{1}{5}\beta D\right]$$

$$\left(G_7\right)_0 = \frac{3}{56}\left[-\frac{7}{8}\beta - \frac{13}{4} - \frac{5}{\beta} - \frac{9}{8\beta^3} - \frac{15}{4\beta^2} + \frac{3}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta - \frac{9}{32}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$- \frac{M^2}{\beta^2}\left[\frac{9}{1792}(\gamma+1)\frac{M^4}{\beta^3} + \frac{2}{7}\gamma\beta D - \frac{1}{28}(\gamma+1)\frac{M^2}{\beta}(45B - 17D) + \frac{1}{7}\beta D\right]$$

$$\left(G_1'\right)_0 = \frac{9}{32}\left[-\frac{5}{2}\beta + 1 + \frac{1}{\beta} + \frac{1}{2\beta^3} + \frac{1}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta + \frac{1}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$\left(G_3'\right)_0 = \frac{9}{32}\left[\frac{11}{2}\beta + 5 - \frac{2}{\beta^2} - \frac{1}{2\beta^3} - \frac{3}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta - \frac{1}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$\left(G_5'\right)_0 = \frac{9}{32}\left[-\frac{25}{6}\beta - \frac{31}{3} - \frac{23}{3\beta} - \frac{3}{\beta^2} - \frac{3}{2\beta^3} + \frac{3}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta - \frac{3}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

$$\left(G_7'\right)_0 = \frac{9}{32}\left[\frac{7}{6}\beta + \frac{13}{3} + \frac{20}{3\beta} + \frac{5}{\beta^2} + \frac{3}{2\beta^3} - \frac{1}{2}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\beta + \frac{3}{8}(\gamma+1)\left(\frac{M^2}{\beta^2}\right)^2\frac{1}{\beta}\right]$$

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TABLE I

VELOCITY AND PRESSURE DISTRIBUTIONS AT THE SURFACE OF A BUMP, $t = 0.10$

θ (deg)	X (Equation (A-9))	Y (Equation (A-9))	q (Equation (A-10))	C_p (Equation (A-11))
0	1.000	0	.8750	.2344
5	.9954	.0001	.8764	.2318
10	.9818	.0005	.8808	.2241
15	.9595	.0017	.8881	.2113
20	.9287	.0040	.8983	.1931
30	.8444	.0125	.9272	.1404
40	.7344	.0266	.9667	.0654
50	.6051	.0450	1.014	-.0292
60	.4625	.0650	1.068	-.1395
70	.3118	.0830	1.117	-.2476
80	.1568	.0955	1.153	-.3299
90	0	.1000	1.167	-.3611

TABLE II

VALUES FOR A, B, C, AND D OBTAINED FROM EQUATION (D-3)

M	β	A	B	C	D
0	1	0.06250	-0.06250	0	0
.10	.99499	.06221	-.06285	.00095	0
.20	.97980	.06160	-.06390	.00400	.00015
.30	.95394	.06169	-.06570	.00982	.00083
.40	.91652	.06499	-.06830	.01990	.00307
.50	.86603	.07788	-.07171	.03751	.00939
.60	.80000	.11822	-.07551	.07057	.02675
.70	.71414	.24561	-.07741	.14211	.07805
.75	.65144	.40177	-.07394	.21371	.13977
.80	.60000	.73343	-.06322	.34482	.26722
.83	.55776	1.11855	-.04583	.48400	.41460
.85	.52678	1.53475	-.02476	.62619	.57315
.90	.43589	4.07940	.12446	1.4241	1.5367
.92	.39192	6.81182	.29917	2.2282	2.5674
.94	.34117	12.9343	.70926	3.9598	4.8722
.96	.28000	31.0576	1.97342	8.8910	11.683
.98	.19900	132.428	9.30328	34.428	49.732

TABLE III

VELOCITY DISTRIBUTION AT THE SURFACE OF A BUMP, $t = 0.10$, ACCORDING TO THE POGGI METHOD

θ (deg)	X'	Y'	X	Y	q (incompressible)	Coefficient of M^2	q (compressible) (Equation (E-17))	
							M = 0.50	M = 0.75
0	1.000	0	1.000	0	0.8750	-0.0540	0.8615	0.8446
5	.9962	.0872	.9954	.0001	.8765	-.0535	.8631	.8464
10	.9848	.1737	.9818	.0005	.8808	-.0521	.8678	.8515
15	.9659	.2588	.9595	.0017	.8881	-.0497	.8757	.8601
20	.9397	.3420	.9287	.0040	.8983	-.0462	.8867	.8723
30	.8660	.5000	.8444	.0125	.9272	-.0359	.9182	.9070
40	.7660	.6428	.7344	.0266	.9667	-.0192	.9619	.9559
50	.6428	.7660	.6051	.0450	1.015	.0039	1.016	1.017
60	.5000	.8660	.4625	.0650	1.068	.0330	1.076	1.086
70	.3420	.9397	.3118	.0830	1.117	.0645	1.133	1.153
80	.1737	.9848	.1568	.0955	1.153	.0795	1.173	1.198
90	0	1.000	0	.1000	1.167	.1002	1.192	1.223

TABLE IV

VALUES OF $(G_1)_0$, $(G_3)_0$, $(G_5)_0$, AND $(G_7)_0$ GIVEN AT END OF APPENDIX F

M	β	$(G_1)_0$	$(G_3)_0$	$(G_5)_0$	$(G_7)_0$
0.50	0.86603	0.10344	-0.79011	1.6714	-0.91449
.75	.65144	2.3075	-2.1109	2.8975	-1.9512
.83	.55776	9.8911	-6.7071	5.9329	-4.0659
.90	.43589	68.849	-42.316	26.227	-19.649

TABLE V

VALUES OF a_1, a_2, AND a_3 OBTAINED FROM EQUATIONS (14)

X \ M	0.50	0.75	0.83	0.90
		a_1		
0	1.7321	2.3026	2.6893	3.4412
.1	1.6974	2.2565	2.6355	3.3724
.2	1.5935	2.1184	2.4742	3.1659
.3	1.4203	1.8881	2.2052	2.8218
.4	1.1778	1.5658	1.8287	2.3400
.5	.86603	1.1513	1.3447	1.7206
.6	.48497	.64473	.75301	.96355
.7	.03464	.04605	.05379	.06882
.8	-.48497	-.64473	-.75301	-.96355
.9	-1.0739	-1.4276	-1.6674	-2.1336
.975	-1.5610	-2.0752	-2.4237	-3.1014
1.0	-1.7321	-2.3026	-2.6893	-3.4412
		a_2		
0	2.2743	5.4350	10.136	19.268
.1	2.0399	4.9892	9.3921	18.076
.2	1.3673	3.7088	7.2538	14.648
.3	.34809	1.7641	4.0015	9.4302
.4	-.86531	-.56063	.10228	3.1662
.5	-2.0593	-2.8678	-3.7900	-3.1035
.6	-2.9594	-4.6459	-6.8342	-8.0405
.7	-3.2325	-5.2746	-8.0105	-10.022
.8	-2.4744	-4.0011	-6.0811	-7.0773
.9	-.23520	.01270	.33212	2.9861
.975	2.7276	5.4013	9.0381	16.723
1.0	4.0063	7.7376	12.825	22.708
		a_3		
0	3.4379	21.877	72.397	437.84
.1	2.5124	18.507	63.333	386.94
.2	.07542	9.4933	38.871	251.93
.3	-2.9313	-2.1171	6.6216	72.788
.4	-5.1915	-12.039	-22.646	-92.533
.5	-5.3952	-15.937	-37.884	-184.67
.6	-2.7710	-11.097	-31.791	-164.59
.7	2.2311	1.4456	-5.9844	-41.555
.8	7.0916	14.256	22.646	92.580
.9	6.2552	10.459	14.506	23.206
.975	-.82172	-20.694	-56.978	-385.70
1.0	-10.033	-39.881	-101.38	-633.92

TABLE VI

VELOCITY DISTRIBUTION FOR A BUMP, $t = 0.10$, CALCULATED BY MEANS OF TABLE V AND EQUATION (13)

	q		
X \ M	0.50	0.75	0.83
0	1.199	1.307	1.443
.1	1.193	1.294	1.421
.2	1.173	1.258	1.359
.3	1.143	1.204	1.267
.4	1.104	1.139	1.159
.5	1.061	1.071	1.059
.6	1.016	1.007	.9752
.7	.9734	.9533	.9193
.8	.9339	.9098	.8865
.9	.8965	.8676	.8444
.975	.8704	.8258	.7910
1.0	.8568	.8072	.7579

TABLE VII

VALUES OF CRITICAL VELOCITY OBTAINED FROM EQUATION (16) AND MAXIMUM VELOCITIES FOR A BUMP, $t = 0.10$, OBTAINED FROM EQUATIONS (17) AND (E-17)

M	q_{max}		q_{cr}
	Iteration method	Poggi method	
0	1.167	1.167	∞
.2	1.171	1.171	4.578
.3	1.176	1.176	3.067
.4	1.185	1.183	2.316
.5	1.199	1.192	1.869
.6	1.222	1.203	1.574
.7	1.265	1.216	1.366
.8	1.371	1.231	1.212
.85	1.523	1.239	1.145
.90	-----	-----	1.093
.95	-----	-----	1.044
1.0	-----	-----	1.000

TABLE VIII

VALUES OF q/q_{cr} FOR A BUMP, $t = 0.10$, WITH $M = 0.83$

X	q	q/q_{cr}
0	1.443	1.230
.1	1.421	1.211
.2	1.359	1.160
.3	1.267	1.080
.4	1.159	.9881
.5	1.059	.9028
.6	.9752	.8314
.7	.9193	.7837
.8	.8865	.7558
.9	.8444	.7200
.975	.7910	.6744
1.0	.7579	.6461

TABLE IX

COMPARISON OF THE PRESSURE DISTRIBUTION AT THE SURFACE OF A BUMP, $t = 0.10$ FOR $M = 0.83$ OBTAINED BY MEANS OF THE ITERATION, THE PRANDTL-GLAUERT, AND THE VON KÁRMÁN METHODS

X	$C_{p,M}$			$C_{p,0}$
	Iteration method (Equation (20))	Prandtl-Glauert method	von Kármán method	
0	-0.9133	-0.6342	-0.7376	-0.3537
.1	-.8677	-.6142	-.7107	-.3426
.2	-.7389	-.5558	-.6337	-.3100
.3	-.5486	-.4665	-.5202	-.2602
.4	-.3294	-.3514	-.3810	-.1960
.5	-.1184	-.2150	-.2257	-.1199
.6	.0514	-.0647	-.0656	-.0361
.7	.1617	.0765	.0752	.0426
.8	.2221	.2062	.1972	.1150
.9	.2868	.3196	.2988	.1783
.975	.4048	.3935	.3620	.2195
1.0	.4708	.4163	.3812	.2322

TABLE X

VALUES OF a_1, a_2, AND a_3 CALCULATED FROM EQUATION (17)

M	a_1	a_2	a_3
0	1.50000	1.50000	1.50000
.2	1.53093	1.58726	1.67024
.3	1.57243	1.71149	1.93357
.4	1.63663	1.92091	2.43349
.5	1.73205	2.27425	3.49200
.6	1.87500	2.91036	5.75400
.7	2.10042	4.22402	12.56611
.8	2.50000	7.76704	43.7829
.85	2.84747	12.52960	112.7760
.90	3.44124	25.7418	446.847
.92	3.82733	38.8963	964.722

TABLE XI

MAXIMUM VALUES OF THE PRESSURE COEFFICIENT $C_{p,M}$ CALCULATED BY MEANS OF EQUATION (20)

	$-(C_{p,M})_{max}$						
M \ t	0.05	0.05 (a)	0.08	0.10	0.12	0.12 (a)	0.15
0	0.16406	0.16406	0.27744	0.36000	0.44856	0.44856	0.59344
.2	.16764	.16773	.28369	.36830	.45913	.45994	.60787
.3	.17248	.17266	.29221	.37964	.47363	.47537	.62779
.4	.18008	.18035	.30570	.39771	.49688	.49963	.66002
.5	.19166	.19188	.32661	.42606	.53375	.53657	.71196
.6	.20971	.20937	.36000	.47200	.59437	.59401	.79907
.7	.24055	.23753	.41994	.55715	.71014	.69006	.97236
.8	.30606	.28925	.56134	.77066	1.0165	.87903	1.4634
.85	.38190	.33621	.74868	1.0737	1.4759	1.0664	2.2485
.90	.59091	.42108	1.3551	2.1252			
.92	.82480	.47966					
M \ t	0.18	0.18 (a)	0.20	0.21	0.22	0.25	0.25 (a)
0	0.75384	0.75384	0.87000	0.93098	0.99396	1.1953	1.1953
.2	.77275	.77541	.89226	.95504	1.0199	1.2274	1.2352
.3	.79899	.80489	.92326	.98859	1.0561	1.2724	1.2903
.4	.84179	.85192	.97408	1.0437	1.1158	1.3470	1.3793
.5	.91186	.92436	1.0581	1.1353	1.2154	1.4732	1.5208
.6	1.0315	1.0403	1.2031	1.2943	1.3891	1.6964	1.7566
.7	1.2782	1.2431	1.5088	1.6326	1.7623	2.1889	2.1973
.8	2.0175	1.6781	2.4539	2.6938	2.9491	3.8118	3.3117

[a]Method of von Kármán.

TABLE XII

CRITICAL AND LIMITING VALUES OF M AND CRITICAL VALUES OF $C_{p,M}$ CALCULATED BY MEANS OF EQUATION (21)

M	$-(C_{p,M})_{cr}$	t	$-C_{p,\theta}$	M_{cr}	M_{lim}	$-(C_{p,M})_{lim}$
0.45	2.76639	0.05	0.16406	0.832	0.890	0.50
.50	2.12953	.08	.27744	.775	.855	.77
.55	1.65519	.10	.36000	.742	.833	.92
.60	1.29190	.12	.44856	.712	.815	1.09
.65	1.00661	.15	.59344	.670	.790	1.33
.70	.77758	.18	.75384	.634	.760	1.58
.75	.59008	.20	.87000	.610	.743	1.74
.80	.43381	.21	.93098	.598	.735	1.82
.85	.30124	.22	.99396	.587	.725	1.90
.90	.18605	.25	1.1953	.558	.698	2.15
.95	.08783					
1.00	0					

TABLE XIII

VALUES OF THE PRESSURE COEFFICIENT $(C_{p,M})_{abs}$ CALCULATED BY MEANS OF EQUATION (22)

M	$-(C_{p,M})_{abs}$	M	$-(C_{p,M})_{abs}$
0.70	2.90508	1.25	.91103
.75	2.53064	1.30	.84230
.80	2.22430	1.35	.78106
.85	1.97022	1.40	.72627
.90	1.75739	1.45	.67705
.95	1.57727	1.50	.63266
1.00	1.42349	1.55	.59250
1.05	1.29115	1.60	.55605
1.10	1.17644	1.65	.52286
1.15	1.07636	1.70	.49256
1.20	.98853	1.75	.46481

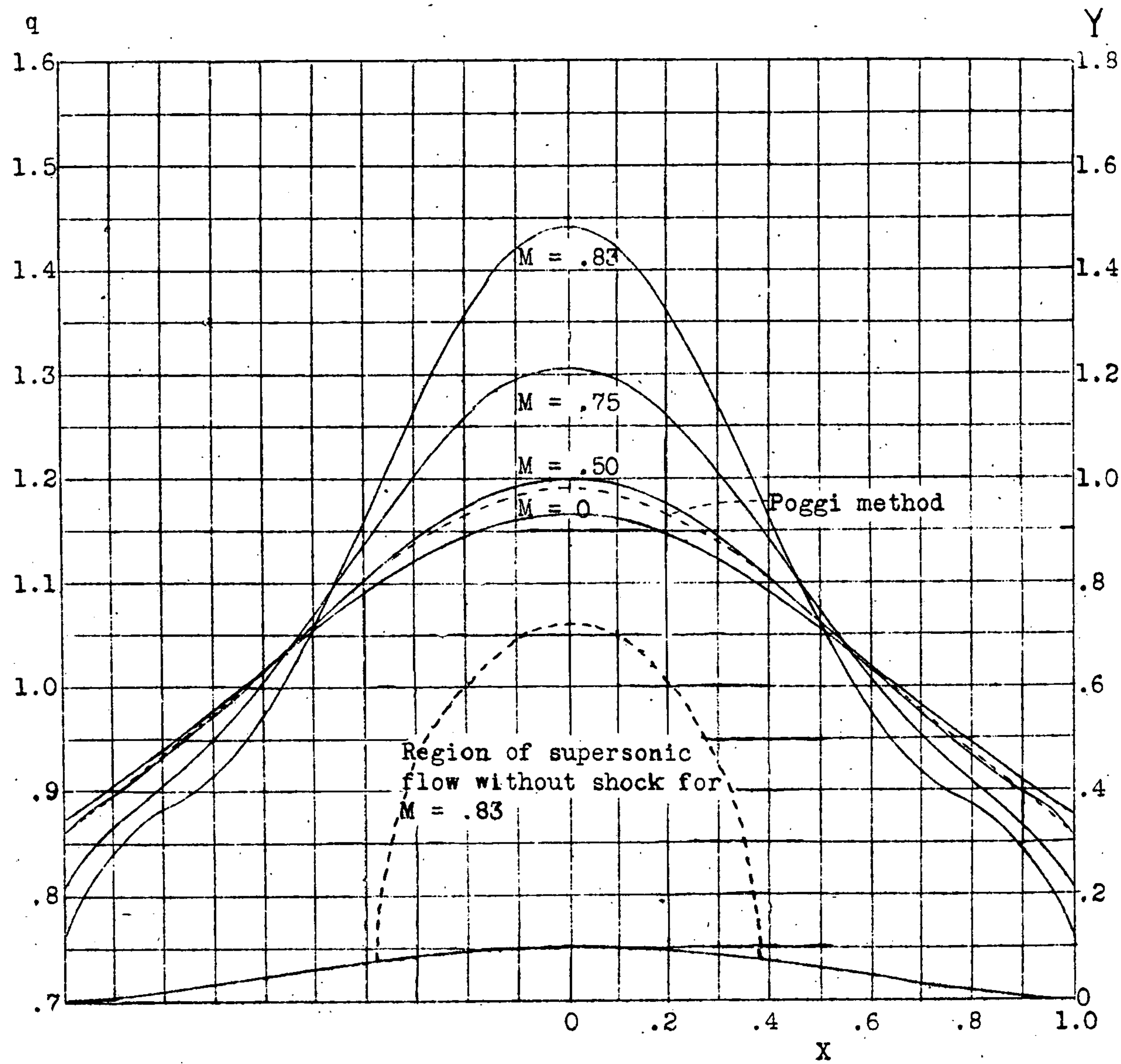

Figure 1.- Velocity distribution at the surface of a bump, t = 0.10, for several values of the Mach number.

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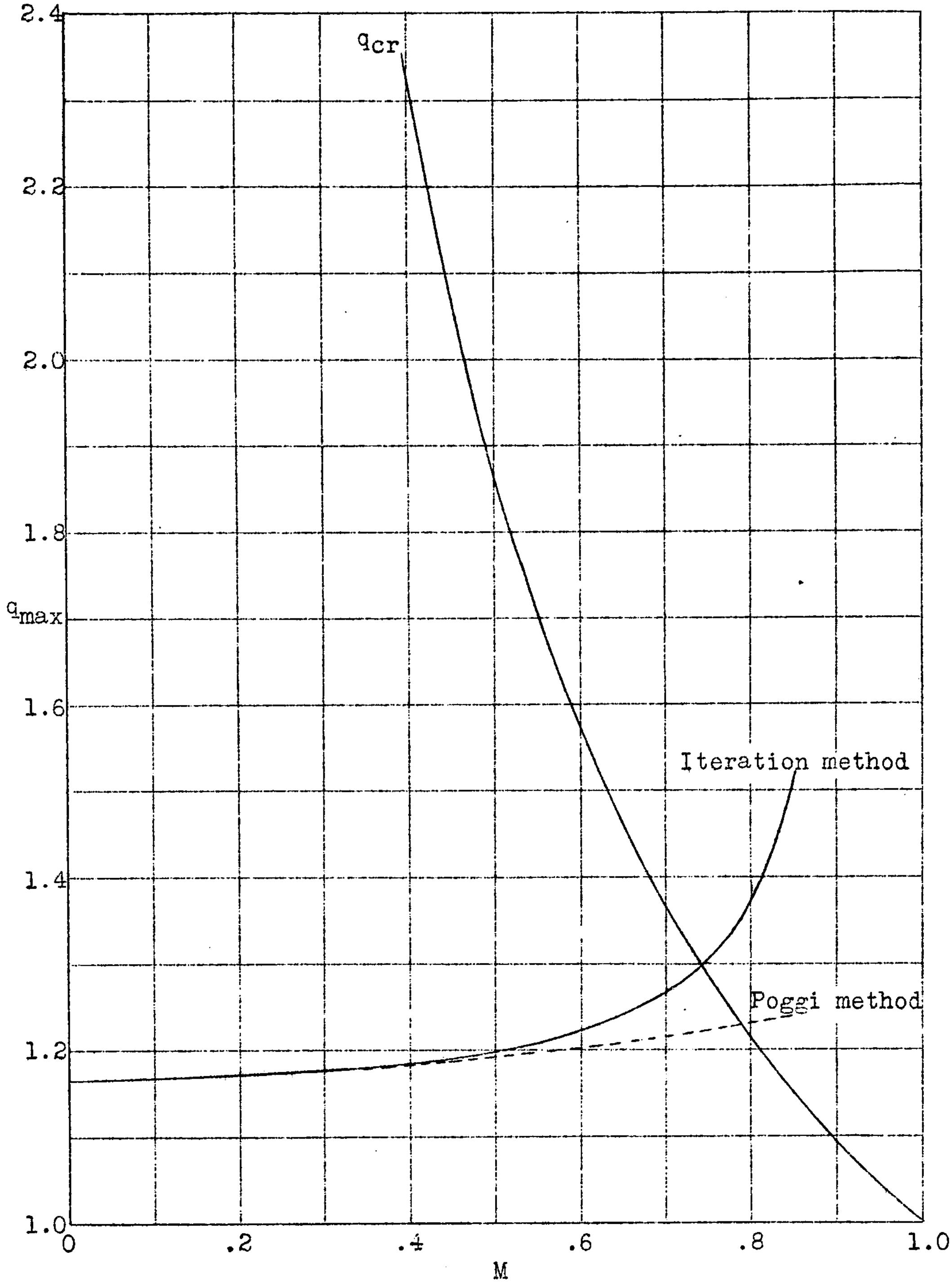

Figure 2.- Maximum velocity at the surface of a bump, t = 0.10.

L-320

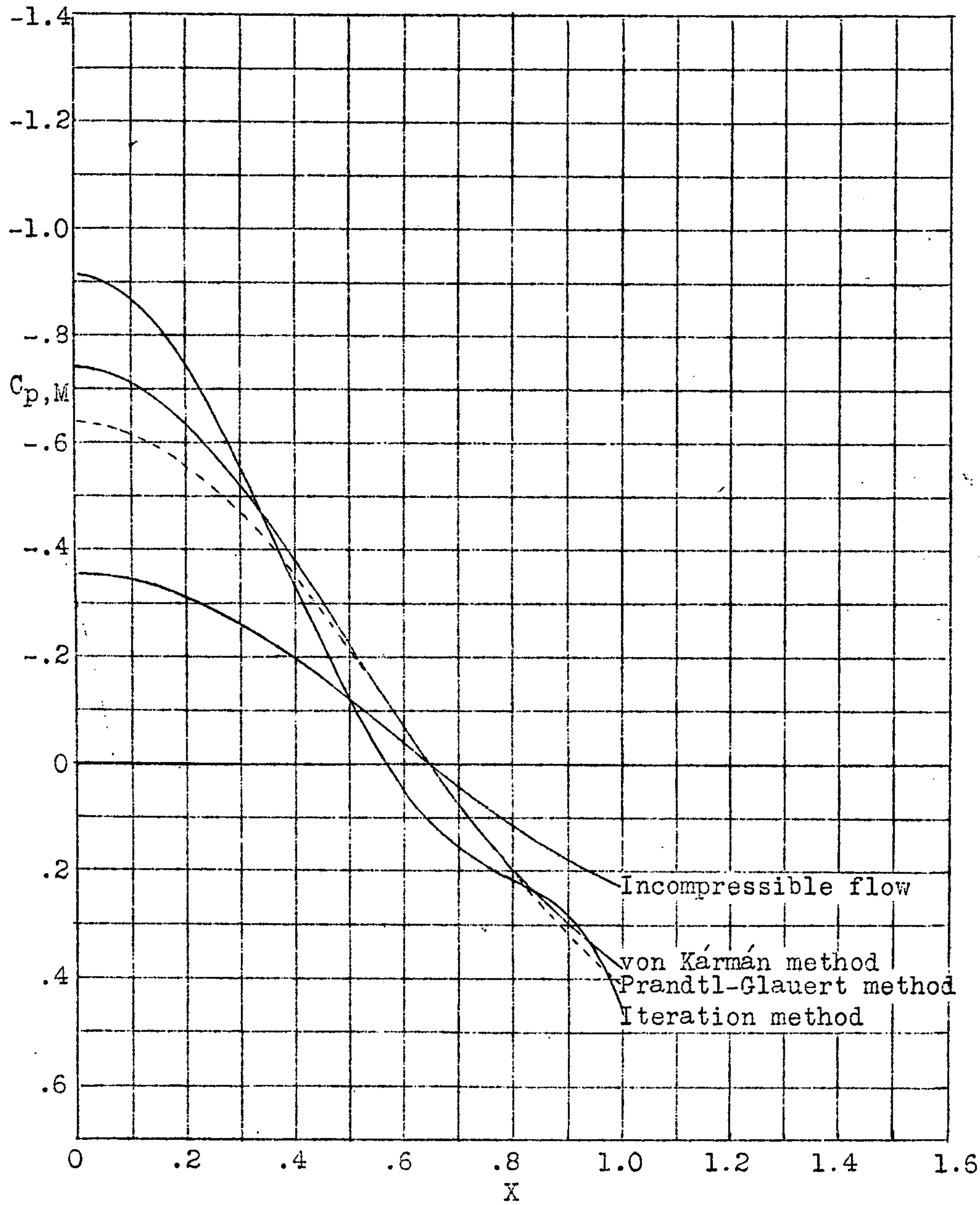

Figure 3.- Pressure distribution at the surface of a bump, t = 0.10, for M = .83.

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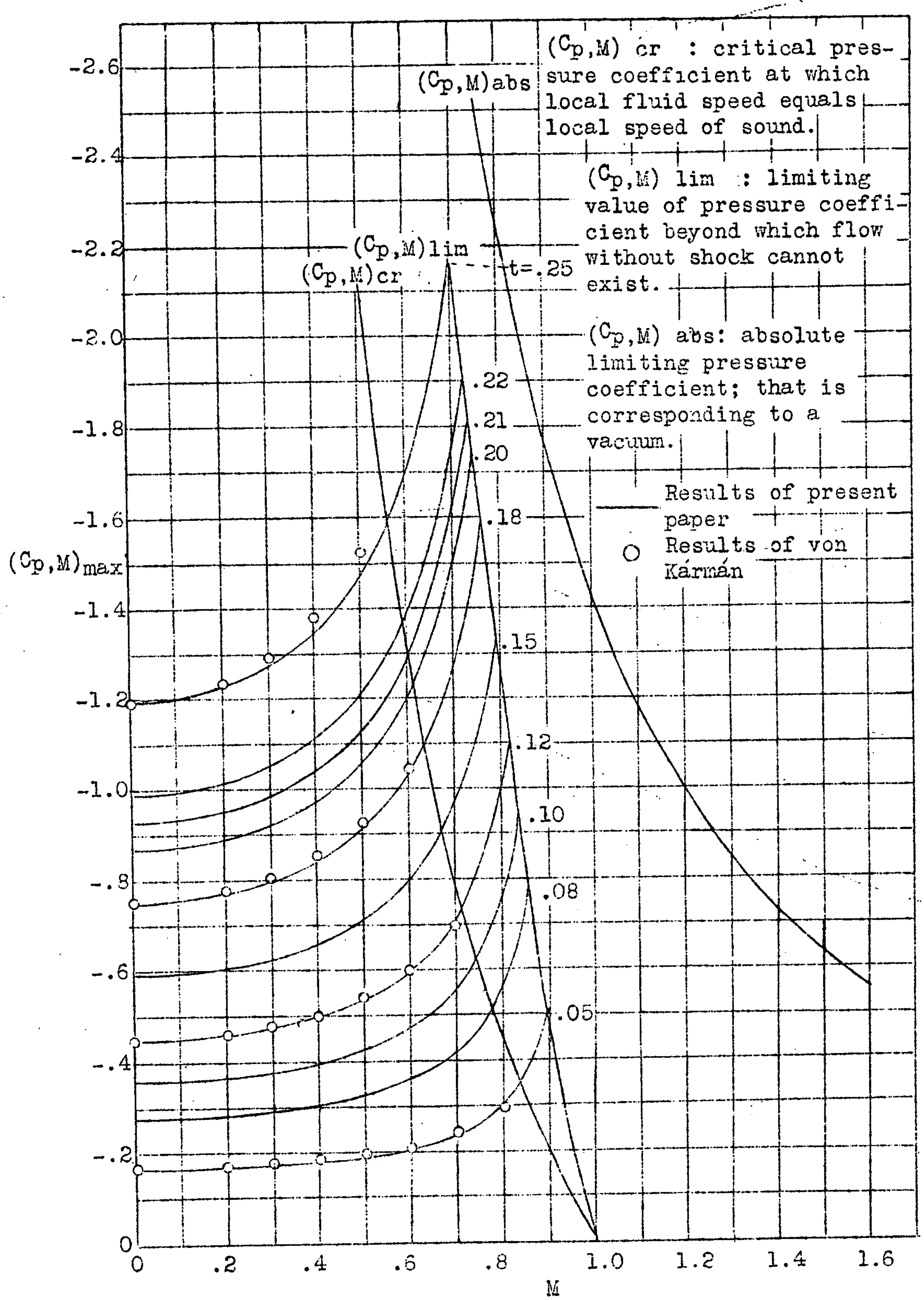

Figure 4.- Maximum pressure coefficient as a function of the Mach number.